Orbital profile and orbit algebra of oligomorphic permutation groups

Conjecture of Macpherson

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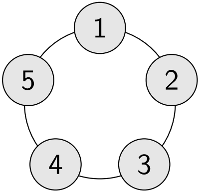
Université Paris-Sud (Orsay)

EJCIM, March 29th of 2018

Age and profile : example on a finite group (1)

Action of the cyclic group $G = C_5$ on the five pearl necklace
$\rightarrow$ induced action on subsets of pearls



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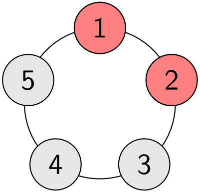

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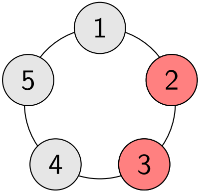

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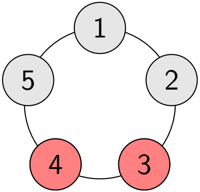

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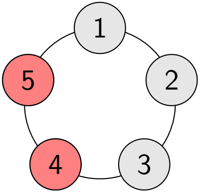

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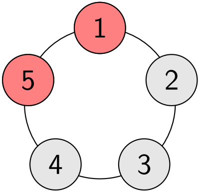

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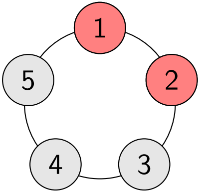

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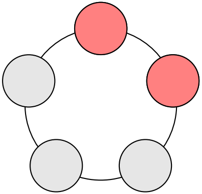

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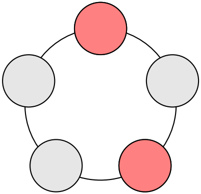

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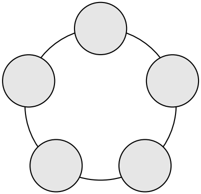

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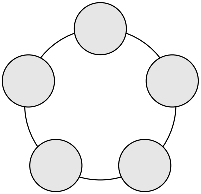

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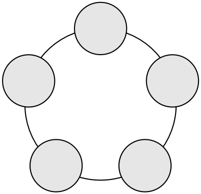

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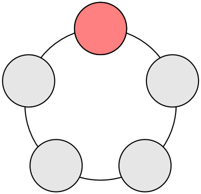

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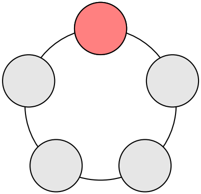

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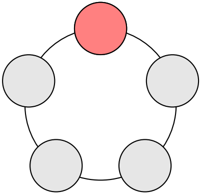

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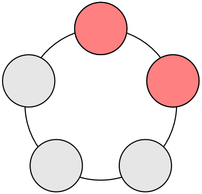

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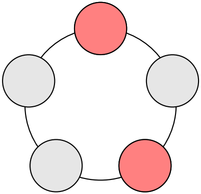

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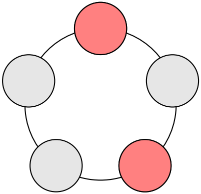

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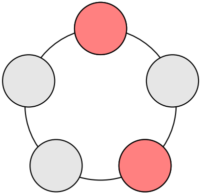

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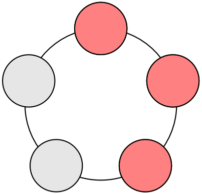

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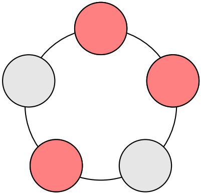

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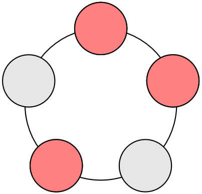

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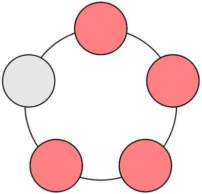

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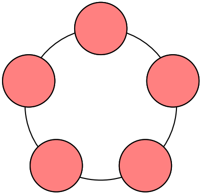

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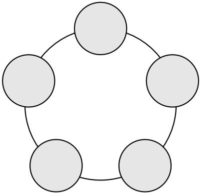

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- Problem : the profile may take infinite values

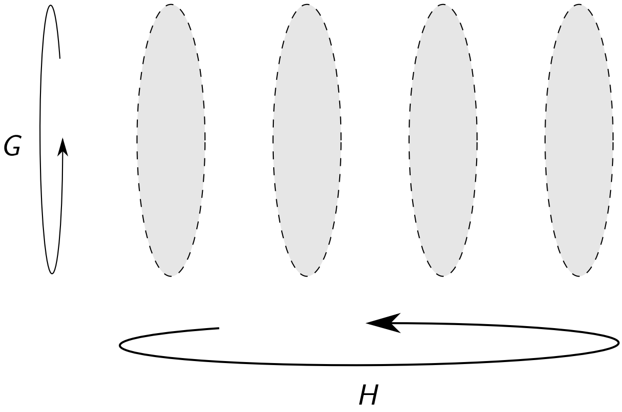
$\rightarrow$ **Oligomorphic permutation groups**:

$$\varphi_G(n) < \infty \quad \forall n \in \mathbb{N}$$

Wreath product of two permutation groups

$G \leq \mathfrak{S}_M$, $H \leq \mathfrak{S}_N$

$G \wr H$ has a natural action on $E = \sqcup_{i=1}^{N} E_i$, with $\text{card} E_i = M$.



Examples

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_\infty$ (action on a denumerable set of copies of $\mathbb{N}$)

  An orbit of degree $n \longleftrightarrow$ a partition of n

  $\varphi_G(n) = \mathscr{P}(n)$, the number of partitions of n

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- $G = \mathfrak{S}_m \wr \mathfrak{S}_\infty$

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  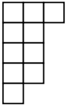

- $G = \mathfrak{S}_\infty \wr \mathfrak{S}_m$

  $\varphi_G(n) = \mathscr{P}_m(n)$, number of partitions into at most m parts

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  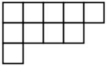

Conjecture of Cameron

Conjecture (Cameron, 70s)

If a profile is bounded by a polynomial it is **quasi-polynomial**:

$$\varphi_G(n) = a_s(n)n^s + \cdots + a_1(n)n + a_0(n),$$

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Note

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})} \implies \varphi_G \text{ quasi-polynomial of degree at most } k-1$$

Graded algebras

Definition: Graded algebra

$A = \oplus_n A_n$ such that $A_i A_j \subseteq A_{i+j}$.

Example

$A = \mathbb{K}[x_1, \ldots, x_m]$ is a graded algebra.

A_n: homogeneous polynomials of degree n

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Proposition

A is finitely generated $\implies$ $\text{Hilbert}(A) = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$

Example

$\text{Hilbert}\left(\mathbb{Q}[x, y, t^3]\right) = \frac{1}{(1-z)^2(1-z^3)}$

A strategy to prove Cameron's conjecture?

- G: an oligomorphic permutation group with polynomial profile
- Find a graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n\geq 0} A_n$ such that

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- Try to show that $\mathbb{Q}\mathcal{A}(G)$ is finitely generated
- Deduce:

$$\mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$$

and thus the quasi-polynomiality of $\varphi_G(n)$

Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

- a commutative connected graded algebra $\mathbb{Q}\mathcal{A}(G) = \oplus_{n\geq 0} A_n$
- $\dim(A_n) = \varphi_G(n)$

Cameron, 1980: the orbit algebra $\mathbb{Q}\mathcal{A}(G)$

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Product?

- Defined on subsets:

$$ef = \begin{cases} e \cup f & \text{if } e \cap f = \emptyset \\ 0 & \text{otherwise} \end{cases}$$

- $o = \{e_1, e_2, \ldots\} \quad \longleftrightarrow \quad e_1 + e_2 + \cdots$

Example of product on a finite case

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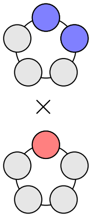

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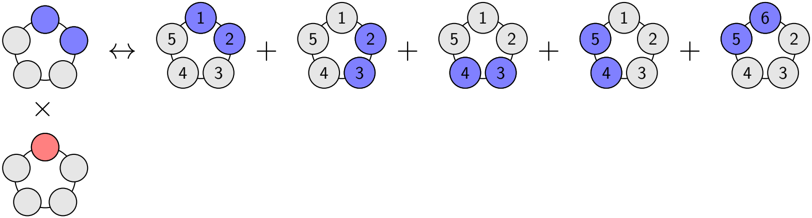

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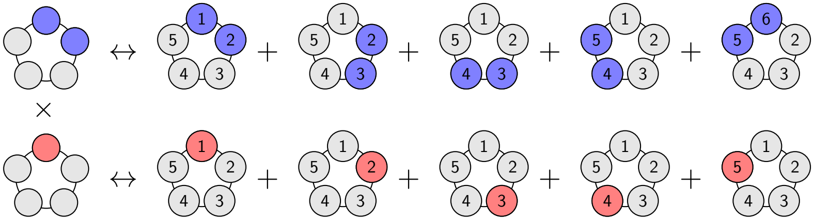

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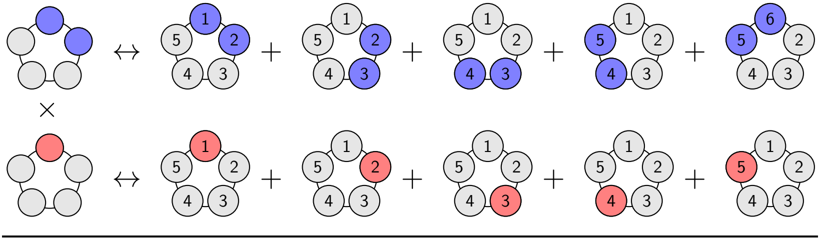

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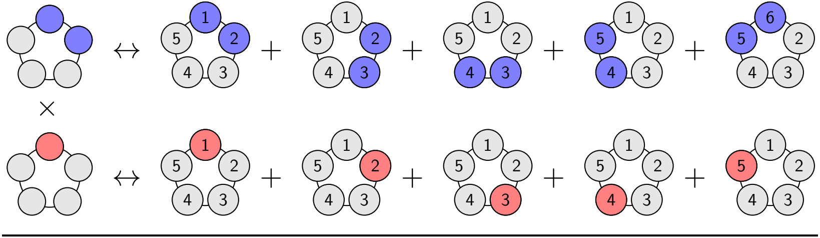

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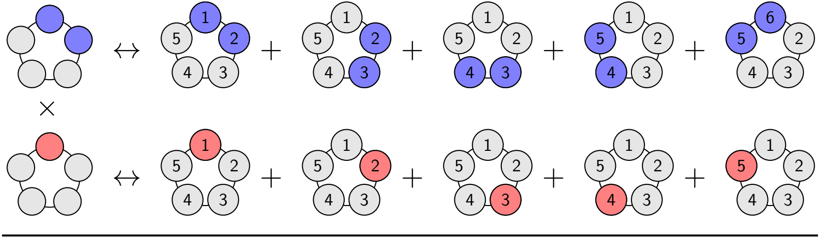

$$= \quad 0 \quad + \quad 0$$

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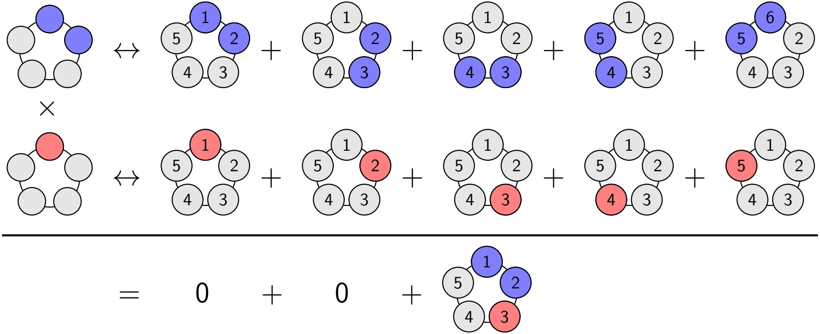

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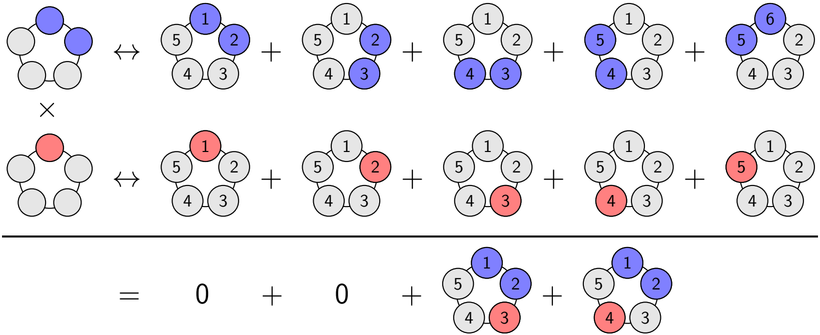

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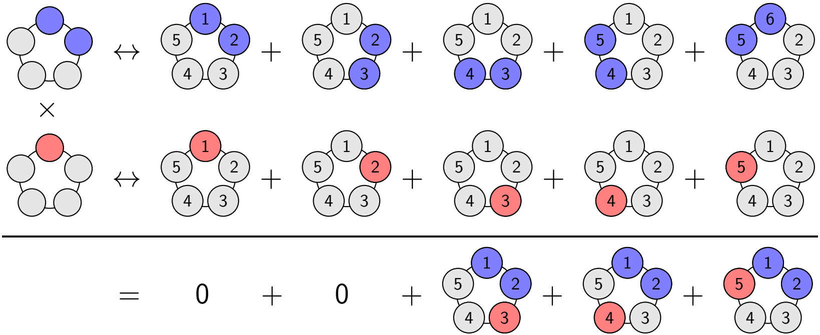

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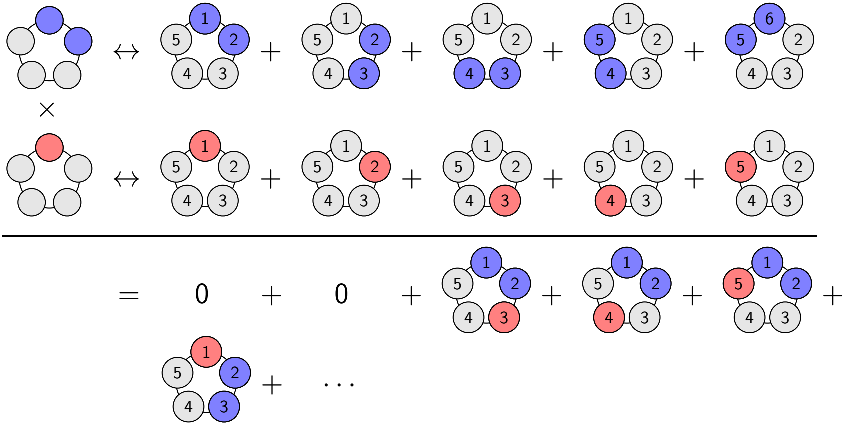

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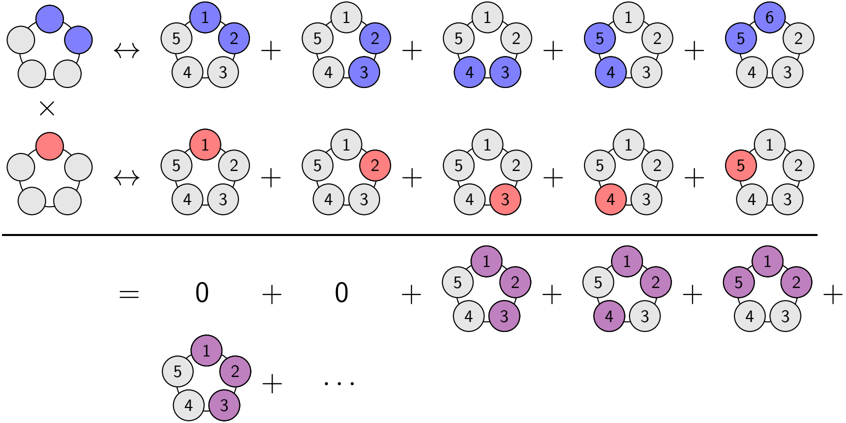

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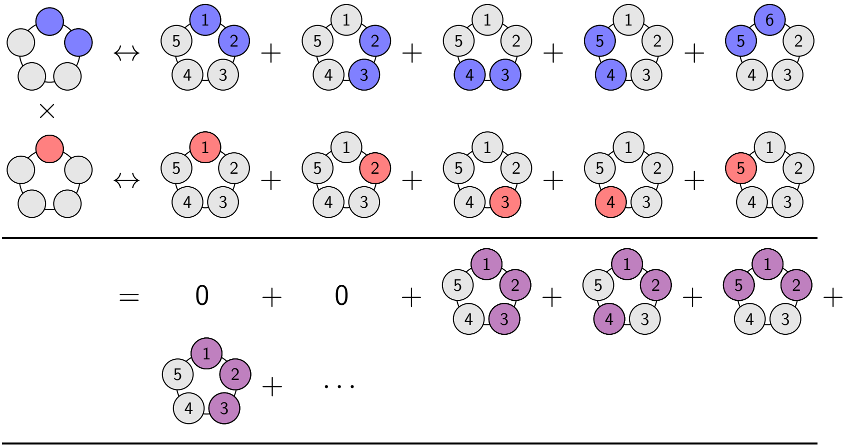

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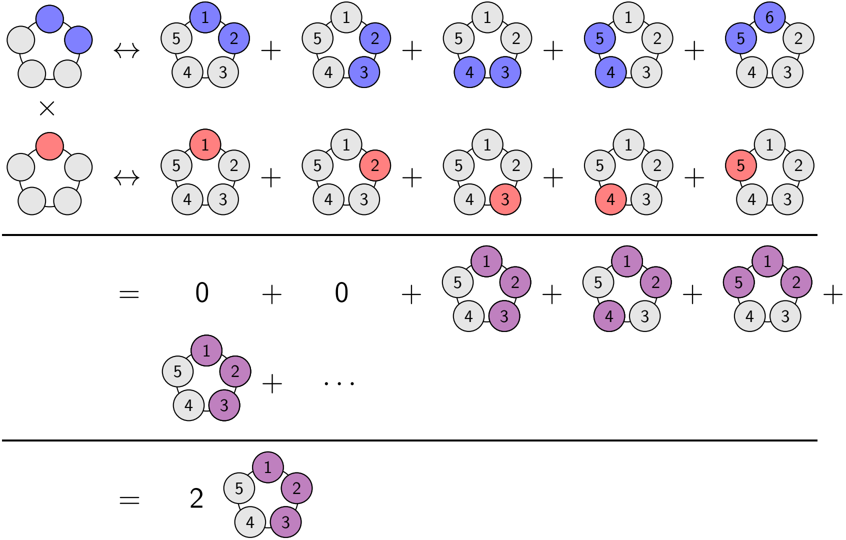

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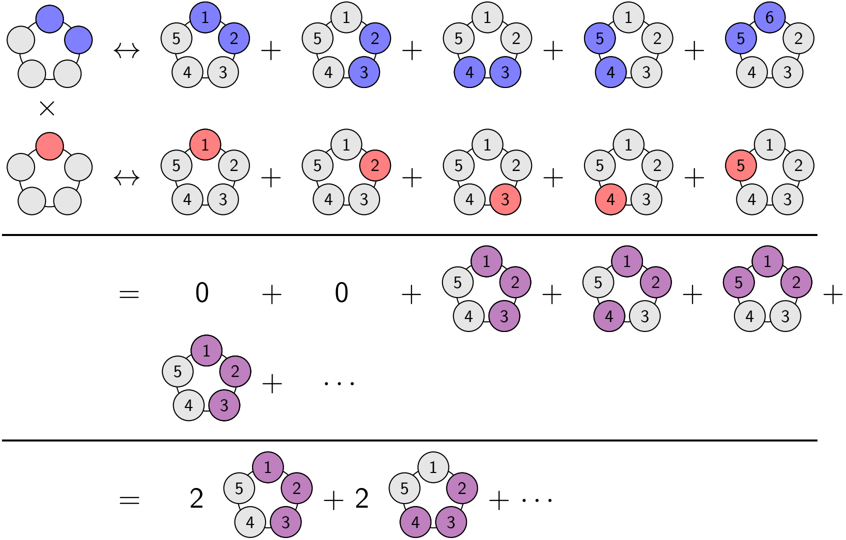

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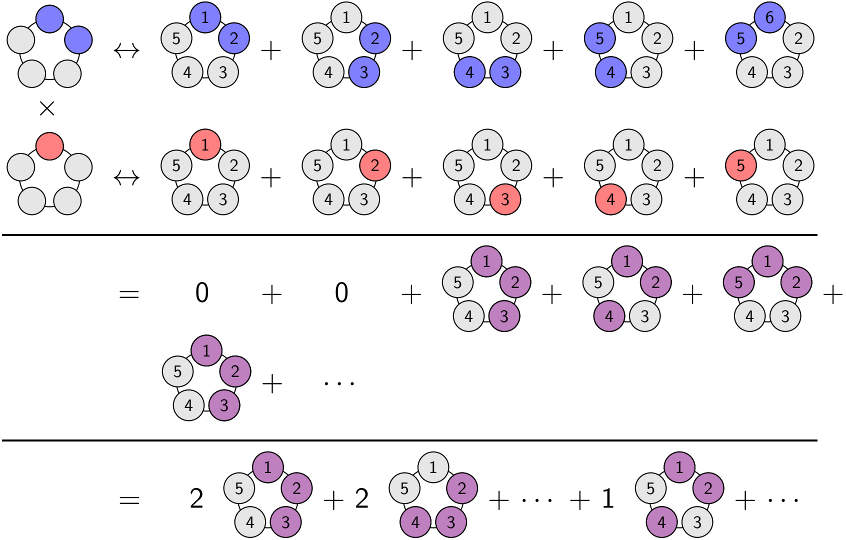

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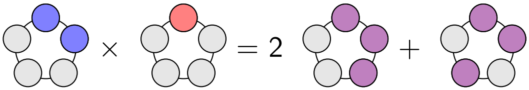
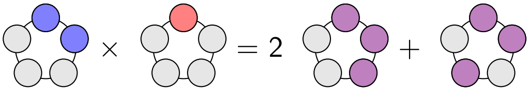


In the end: 

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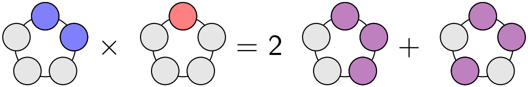

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$\longrightarrow$ **The orbit algebra of a permutation group**

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If $G = \mathfrak{S}_\infty$, $\varphi_G(n) = 1$ for all n, and $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x]$.

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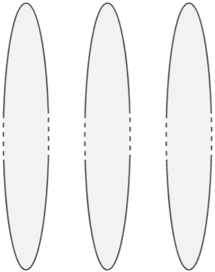

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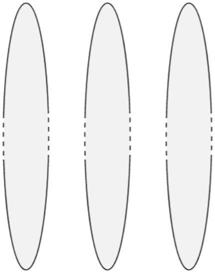

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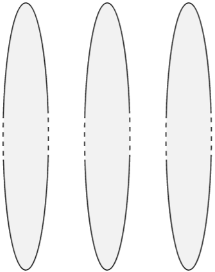

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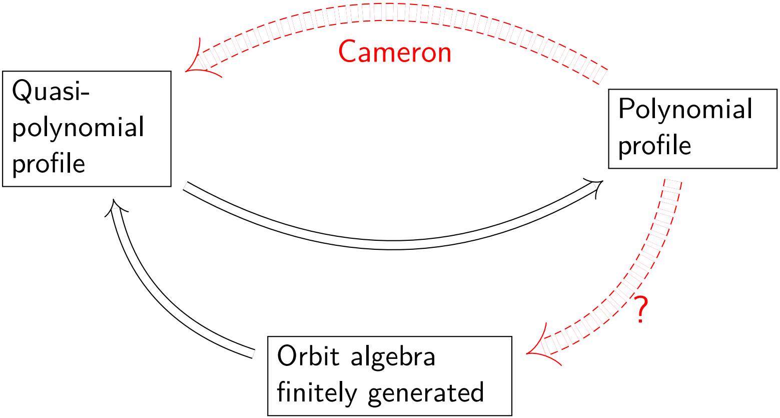
More generally, for H subgroup of $\mathfrak{S}_m$, $\mathbb{Q}\mathcal{A}(\mathfrak{S}_\infty \wr H) = \mathbb{K}[x_1, \ldots, x_m]^H$, the algebra of invariants of H

Overview and conjecture of Macpherson



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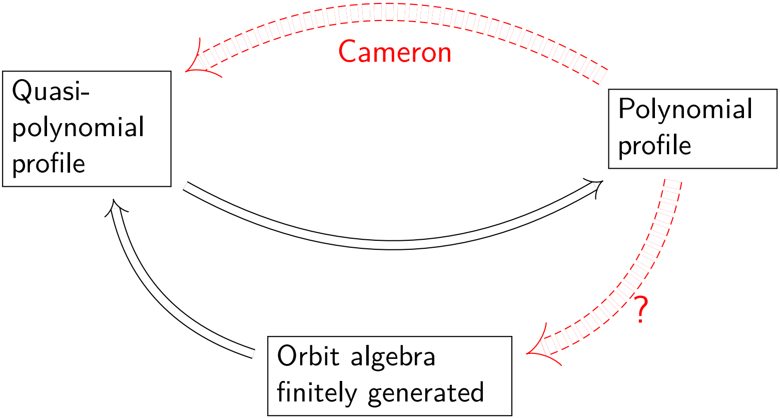
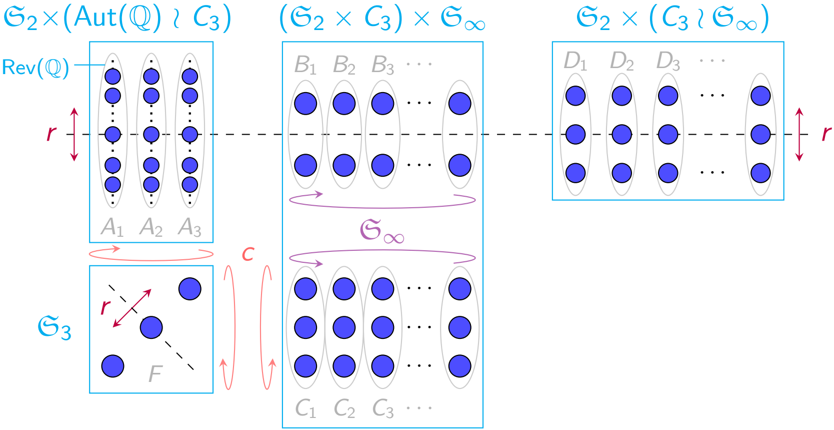


Conjecture (Macpherson, 1985)

Profile of G polynomial $\Longleftrightarrow$ $\mathbb{Q}\mathcal{A}(G)$ finitely generated

A typical group with profile bounded by a polynomial



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$\longrightarrow$ *The conjectures of Macpherson and Cameron hold !*

Thank you for your attention !

Context

- G permutation group of a countably infinite set E
- Profile φ_G: counts the orbits of finite subsets of E
- **Hypothesis** : $\varphi_G(n)$ bounded by a polynomial
- Conjecture (Cameron) : quasi-polynomiality of φ_G
- Conjecture (Macpherson) : finite generation of the orbit algebra

Results

- Both conjectures hold
- The orbit algebra is a Cohen-Macauley algebra

Question

- On what algebra ? What about higher growths ?

Finite index subgroups

Theorem

Let H be a finite index subgroup of G.

- The profiles of G and H are asymptotically equivalent
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Application : reduction of Macpherson's conjecture

Without loss of generality, we may assume for instance that G has no finite orbit.

But there will be more...

Block systems

Definition : Block system

Partition of E such that each part is globally mapped onto another one (or itself) by every element of G

(see previous examples)

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→ The groups we are interested in have a constantly equal to 1 profile or have a block system.

The complete primitive groups

Theorem (Classification, Cameron)

There are exactly 5 complete groups of constantly equal to 1 profile.

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- $\mathrm{Aut}(\mathbb{Q})$: automorphisms of the rational chain (increasing functions)
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- $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$, preserving the circular order
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Well known, nice groups.
In particular, their orbit algebra is finitely generated.

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If G is transitive on a block system, it either has a finite system of infinite blocks or an infinite system of finite blocks of the same size.

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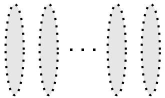

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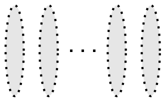

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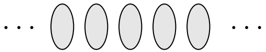

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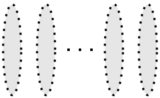

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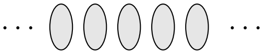

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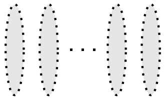

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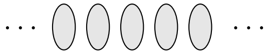

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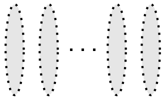

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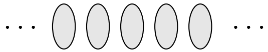

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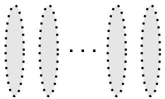

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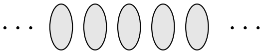

- $B(G)$ $\rightarrow$ action on the blocks is primitive

Actually, G acts on the blocks as $\mathfrak{S}_\infty$

Synchronization

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$E_1 \sqcup E_2$, $\quad G_{|E_1} = G_1, \; G_{|E_2} = G_2$

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Lemma

The five complete groups of profile 1 have at most one non trivial normal subgroup.

$\rightarrow$ very few possibilities

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Evaluated by $G_1/N_1 = G_2/N_2$, where $N_i = \mathrm{Fix}(G, E_j)_{|E_i}$

Lemma

The five complete groups of profile 1 have at most one non trivial normal subgroup.

$\rightarrow$ very few possibilities

Example

If $G_1 = G_2 = \mathfrak{S}_\infty$, the actions are either independant or totally synchronized. One may assume safely, for our purposes, the same about the other four groups.

Application to the canonical block system

Works on orbits of blocks → essentially independant in $B(G)$

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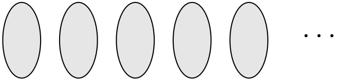
- Wreath products $\rightarrow$ OK
- Direct products $\rightarrow$ OK
- General case ?

The "hard case" : transitive block system of finite blocks



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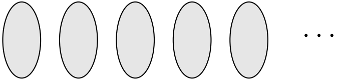

Definition : Tower of G

H_0 H_1 H_2 ... where H_i is the restriction to the block $i+1$ of the subgroup of G that stabilizes all the blocks and acts trivially on the first i blocks.

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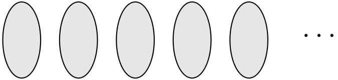

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Same tower $\Longrightarrow$ Same orbit algebra

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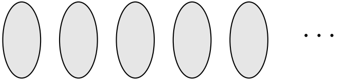

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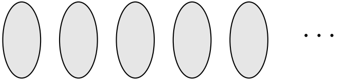

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Proposition 2

The tower of G must be of shape : $H_0\ H\ H\ H\ \dots$
Thus, G has the same orbit algebra as $\frac{H_0}{H} \times H \wr \mathfrak{S}_\infty$,
which is of finite index over $H \wr \mathfrak{S}_\infty$.

The "hard case" : transitive block system of finite blocks

Sketch of proof.

1. Finite case of four blocks only :
   G has tower $H_0\ H_1\ H_2\ H_3 \Rightarrow H_1 = H_2$

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- Restrictions to orbits of finite blocks may be thought of as wreath products for the sake of proving the conjecture
- Solves the issue of possible finite synchronizations between different orbits of blocks

Stronger result : Cohen-Macauley algebra

- Finite generation of the orbit algebra $\Rightarrow \mathcal{H}_G = \frac{P(z)}{(1-z^{d_1})\cdots(1-z^{d_k})}$

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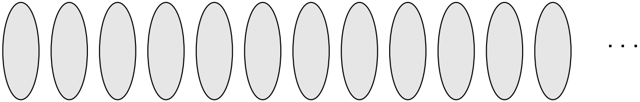
- Case of Cohen-Macauley algebras (free finite module over a free finitely generated algebra) : $\exists\ P(z)$ with positive coefficients
- Once again, it is possible to adapt a proof of invariant theory to obtain that the orbit algebra is indeed a Cohen-Macauley algebra

Direct product in the case of finite blocks

"Speak, friend..."



Direct product in the case of finite blocks

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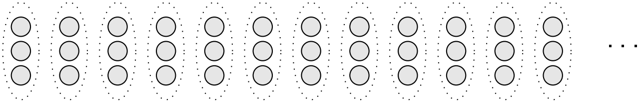
Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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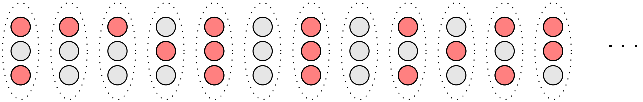

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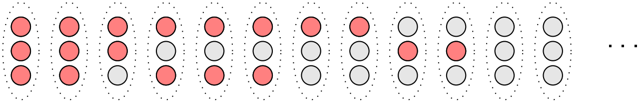

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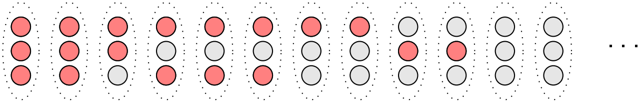

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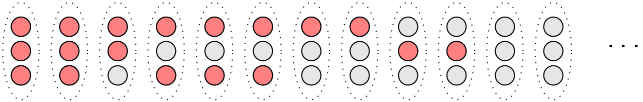

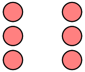

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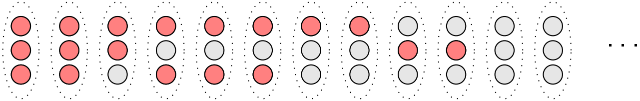

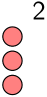

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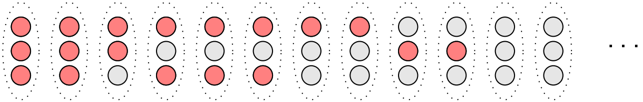

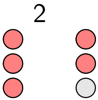

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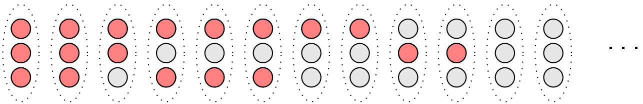

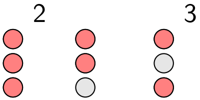

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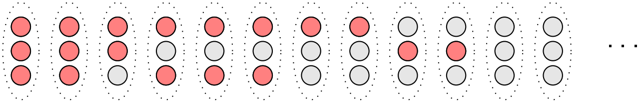

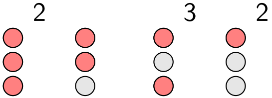

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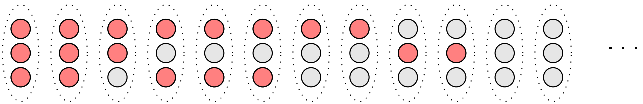

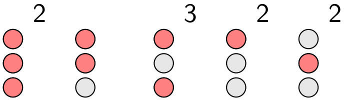

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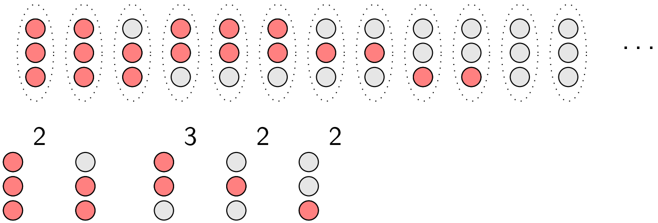

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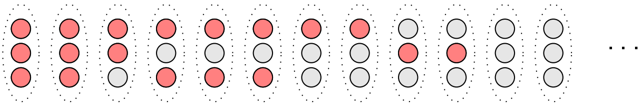

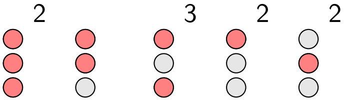

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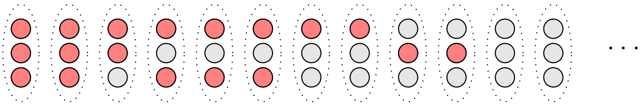

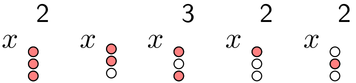

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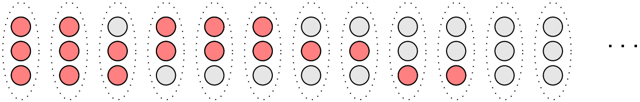

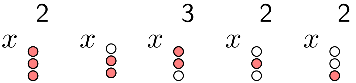

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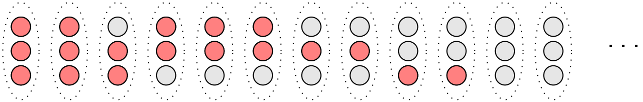

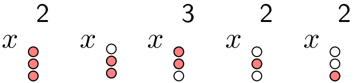

$\rightarrow$ C_3 acts on monomials

Direct product in the case of finite blocks

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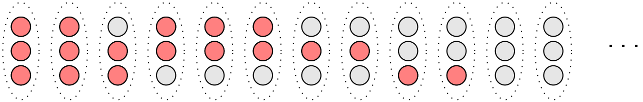

$G' = C_3$ acting on (non empty) subsets

$\mathbb{K}[\ x\ ]^{G'} \quad \longleftrightarrow \quad$ Orbit algebra of $C_3 \times \mathfrak{S}_\infty$?

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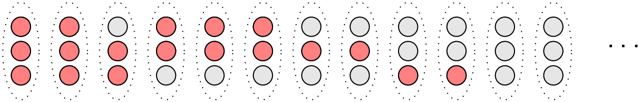

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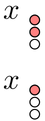

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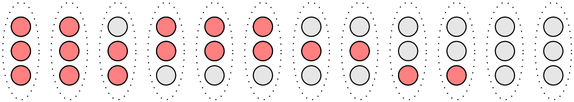

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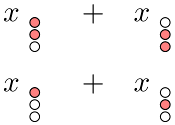

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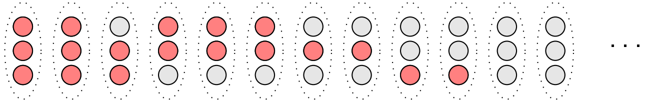

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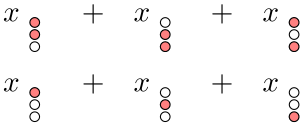

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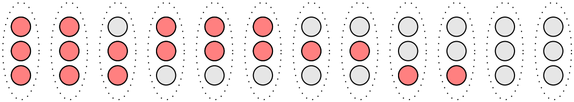 . . .

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O(x )

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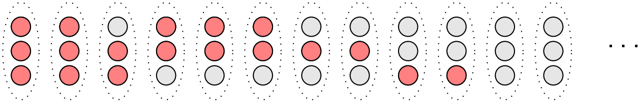

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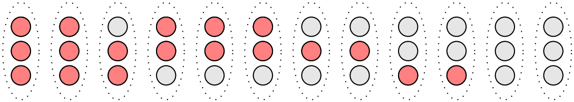

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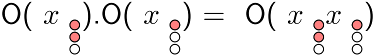

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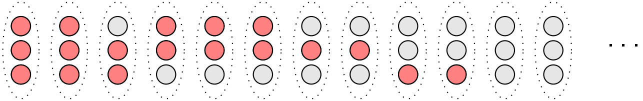

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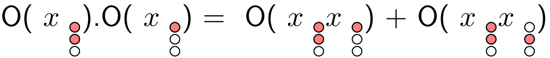
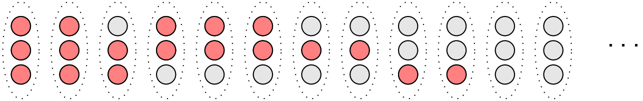


Direct product in the case of finite blocks

"Speak, friend..."

Example 3

$C_3 \times \mathfrak{S}_\infty$ acting on blocks of size 3



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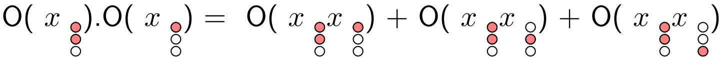

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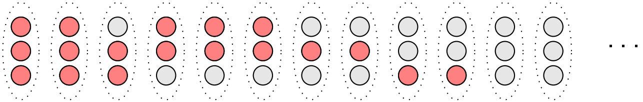

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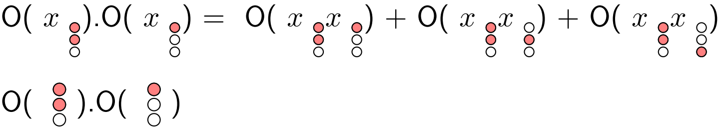

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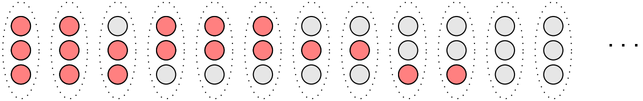

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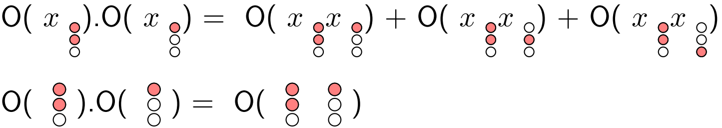

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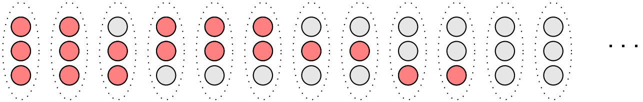

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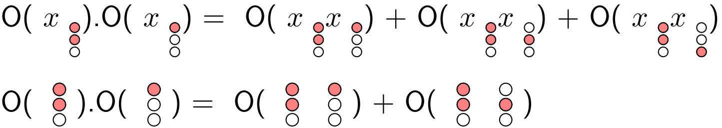

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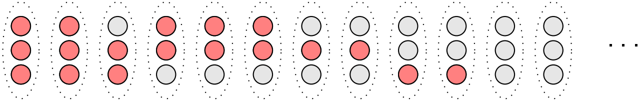

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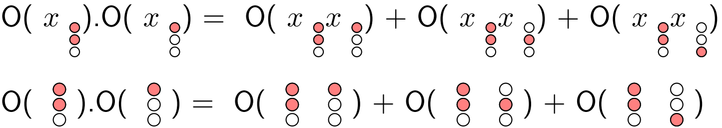

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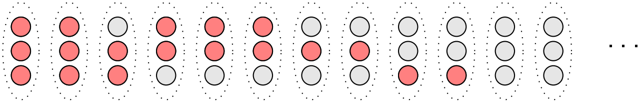

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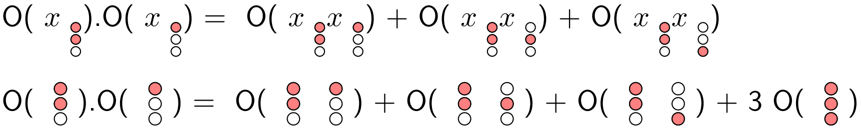
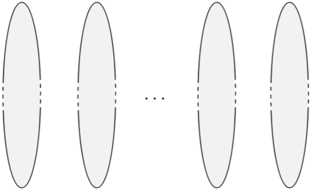


Examples of orbit algebras (2)

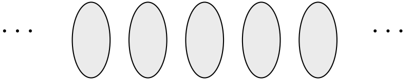
More generally, for H subgroup of $\mathfrak{S}_m$:

- $G = \mathfrak{S}_\infty \wr H$:
  $\mathbb{Q}\mathcal{A}(G) = \mathbb{K}[x_1, \ldots, x_m]^H$, the algebra of invariants of H

  $\mathbb{Q}\mathcal{A}(G)$ is finitely generated by Hilbert's theorem.



- $G = H \wr \mathfrak{S}_\infty$:
  $\mathbb{Q}\mathcal{A}(G) =$ the free algebra generated by the age of H



The "hard" case : case of four blocks

Lemma to prove

G has tower $H_0\ H_1\ H_2\ H_3 \Rightarrow H_1 = H_2$

Lemma

In the general case :
$Fix_G(B_1, \dots, B_n)$ acts on the remaining blocks as $\mathfrak{S}_\infty$
(due to the absence of normal subgroup of finite index of $\mathfrak{S}_\infty$).

Proof.

An element $s \in G$ stabilizing the blocks $\leftrightarrow$ a quadruple
$g \in H_1 \quad \rightarrow \quad \exists\ (1, g, h, k), \quad h, k \in H_1$.
Let σ be an element of G that permutes the first two blocks and fixes the other two.
Conjugation of x by σ in $G \quad \rightarrow \quad y = (g', 1, h, k)$
Then: $x^{-1}y = (g', g^{-1}, 1, 1)$
By arguing that the tower does not depend on the ordering of the blocks, g^{-1} and therefore g are in H_2.