Efficiency of Local Search for Geometric Combinatorial Optimisation

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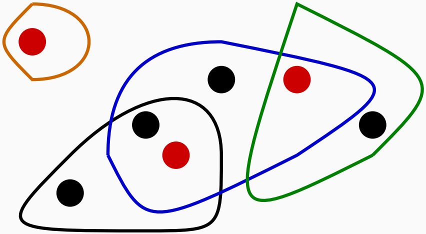
The General HITTING SET Minimisation Problem is Hard

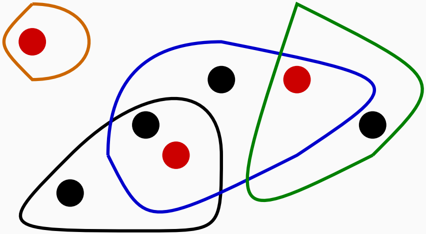


MINIMUM HITTING SET

Given (up to) n points and sets, compute a smallest set of points hitting all sets.

Dashed hopes: this is fairly hard

- ✗ NP-hard (one of Karp's 21)
- ✗ cannot do better in polynomial time than a $\frac{\log n}{2}$-approximation unless $P = NP$



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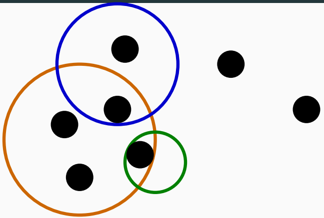

Now the input sets are disks!

- × still NP-hard
- × "likely" no $(1+\epsilon)$-approximation in $f(\epsilon) \cdot n^c$ (W[1]-hard)
- ✓ $(1+\epsilon)$-approximation in $n^{O(1/\epsilon^2)}$ [Mustafa & Ray '10]
  = a "polynomial-time approximation scheme"

Two questions for today

- How/why does this PTAS work?
- Why $1/\epsilon^2$ rather than $1/\epsilon$?

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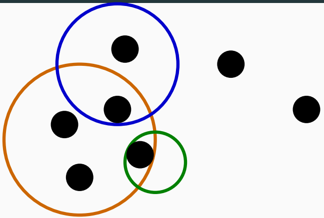

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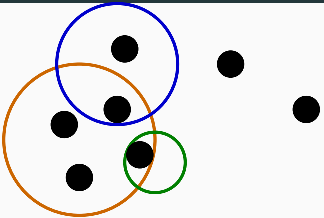

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The Local Search Heuristic (Is Surprisingly Efficient)

λ-local optimum

Any hitting set that is smallest within Hamming distance 2λ: cannot be improved by dropping λ elements and adding $\lambda - 1$.

A local optimum can always be found in time $n^{O(\lambda)}$. (How?)

$\left(1+\frac{c}{\sqrt{\lambda}}\right)\mathrm{Opt}$ in time $n^{O(\lambda)}$ gives $(1+\epsilon)\mathrm{Opt}$ in time $n^{O(\epsilon^{-2})}$.

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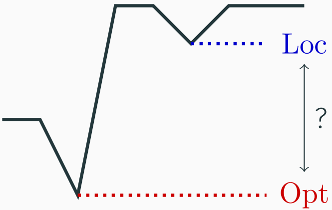

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For our hitting set problem,

$$\frac{\text{Loc}}{\text{Opt}} \leq 1 + \frac{c}{\sqrt{\lambda}}$$

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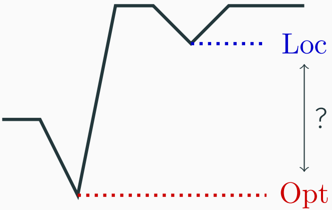

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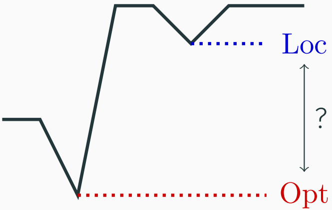

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What Goes into the Lemma – 1

Take:

- $\mathcal{L}$ a λ-locally optimal hitting set: $|\mathcal{L}| = \mathrm{Loc}$, ($\lambda$ large)
- $\mathcal{O}$ a globally optimal hitting set: $|\mathcal{O}| = \mathrm{Opt}$.

Bipartite graph $\mathcal{G}$ on $\mathcal{L} \cup \mathcal{O}$: edge xy *iff* some disk of $\mathbb{R}^2$ contains only x and y. (Delaunay graph)

Claim

In particular, every input disk contains an edge! So for any $L \subseteq \mathcal{L}$, the set $(\mathcal{L} \setminus L) \cup N_{\mathcal{G}}(L)$ is also a solution.

Two key properties of $\mathcal{G} = (\mathcal{L} \cup \mathcal{O}; E)$

1. $\mathcal{G}$ is planar, (can be drawn without crossing in $\mathbb{R}^2$)
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What Goes into the Lemma – 2

Recall: two key properties of $\mathcal{G} = (\mathcal{L} \cup \mathcal{O}; E)$

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2. "λ-expanding": for any $L \subseteq \mathcal{L}$, if $|L| \leq \lambda$ then $|N_{\mathcal{G}}(L)| \geq |L|$.

Theorem (Chan & Har-Peled '09, Mustafa & Ray '10)

Any bipartite graph that satisfies these two properties has

$$|\mathcal{L}| \leq \left(1 + \frac{c}{\sqrt{\lambda}}\right) |\mathcal{O}|.$$

Proof ingredients

Planar $O(\sqrt{n})$ separators (Lipton & Tarjan '79), iterated (Frederickson '86). Then apply (2) on each piece.

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A Theorem on Planar, Locally Expanding Graphs

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Manage to replace $\sqrt{\lambda}$ with λ and our running time goes from $n^{O(1/\epsilon^2)}$ to $n^{O(1/\epsilon)}$... but...

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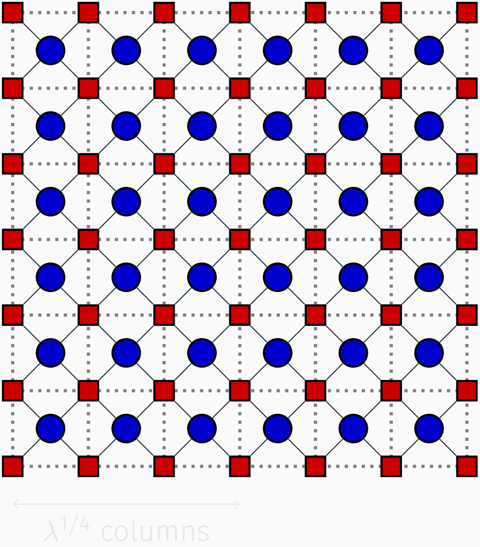
A (Roughly) Balanced Locally-Expanding Bipartite Planar Graph



1. Start with a bipartite grid.
   $|\mathcal{L}| \simeq |\mathcal{O}|$,
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2. Periodically duplicate 1 in $\sqrt{\lambda}$ blue vertex.
3. This new graph is $\Theta(\lambda)$-expanding.
   (Proof is simple but not obvious).
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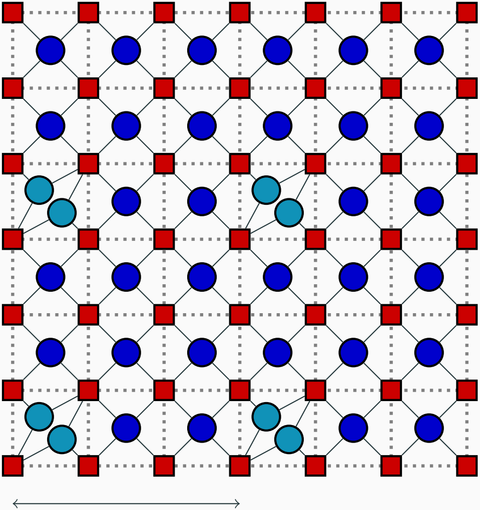

$\lambda^{1/4}$ columns

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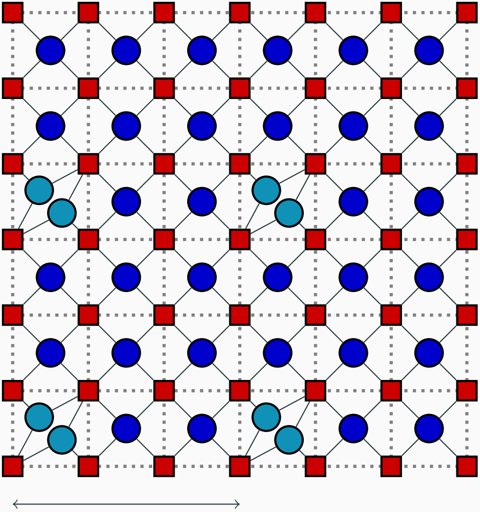

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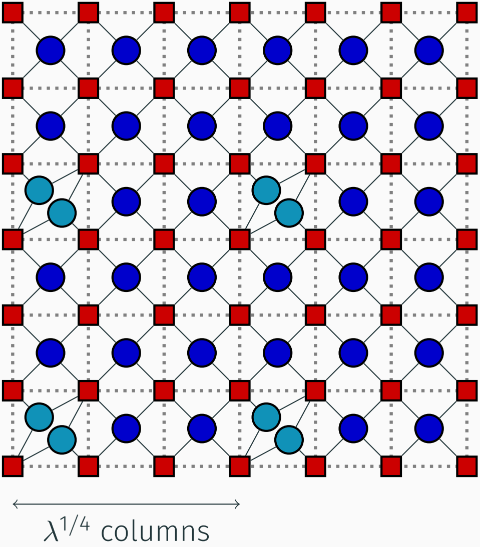

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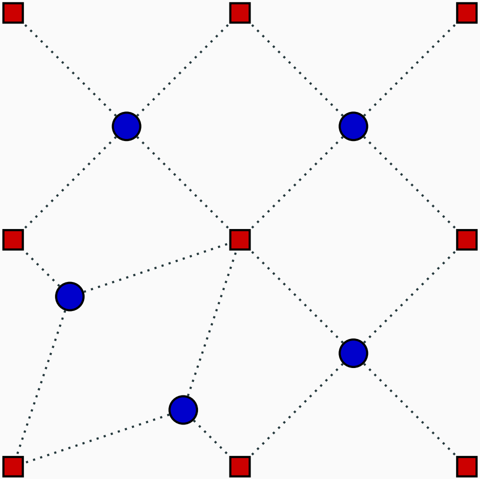
Building Instances of Minimum Hitting Set that Attain the Gap



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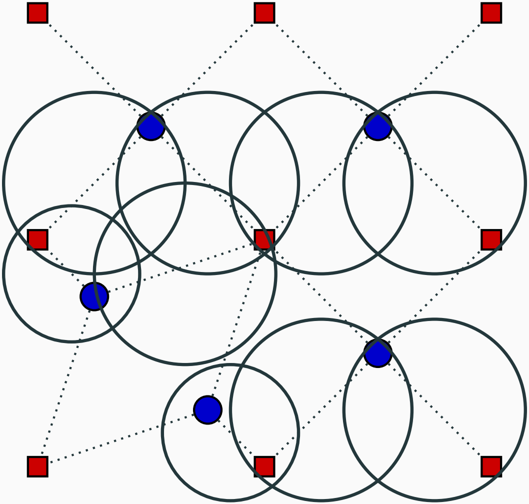

Consequences and Extensions

What we know

- We construct "bad" instances of Hitting Set for Disks where local search radius has to be $\Theta(1/\epsilon^2)$.
  Also: Independent Set, Dominating Set, Set Cover,...
- Extensions to graphs with separators in $O(n^{1-1/d})$: local search radius $\Theta(1/\epsilon^d)$.
- Results for small λ: planar λ-expanding graphs have

$$\lambda = 3: \quad |\mathcal{L}| \leq 8|\mathcal{O}| \quad \text{[Bus et al. '15]}$$

$$\lambda = 4: \quad |\mathcal{L}| \leq 4|\mathcal{O}| \quad \text{[Antunes et al. '17]}$$

Question

What is the correct bound on $|\mathcal{L}|/|\mathcal{O}|$ for $\lambda = 5$?