Continued Logarithm Algorithm. A probabilistic study

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Work with

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$$\frac{13}{31} = \cfrac{2^{-1}}{1 + \cfrac{2^{-2}}{1 + \cfrac{2^{-1}}{1 + \cfrac{2^{0}}{1 + \cfrac{2^{-1}}{1}}}}}$$

EJCIM, **Nancy**, March, 2018.

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The origins

In Hakmem *Gosper* writes

"*There is a mutation of continued fractions, which I call continued logarithms, which have several advantages over regular continued fractions, especially for computational hardware.*

(...) *The primary advantage is the conveniently small information parcel.*

(...) *the continued logarithm of Avogadro's number begins with its binary order of magnitude, and only then begins the description equivalent to the leading digits – a sort of recursive version of scientific notation.*"

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The continued logarithm algorithm computes the *odd* $\gcd$ and

- involves quotients that are powers of 2.
- seems simple and efficient.
- let us see an example!

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Example. Let us find $\gcd(31, 13)$.

a	q	p	$2^a p$	r
1	31	13	26	5
2	26	5	20	6
1	20	6	12	8
0	12	8	8	4
1	8	4	8	0

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Remark.

- We ended up with $(8, 0)$, what is the $\gcd$? $\Rightarrow$ odd $\gcd = 1$.

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Average number of steps K and shifts S satisfy

$$E_N[K] \sim k \log N \,, \qquad E_N[S] \sim \tfrac{\log 3 - \log 2}{2\log 2 - \log 3} \, E_N[K]$$

for an *explicit constant* $k \doteq 1.49283\ldots$ given by

$$k = \frac{2}{H}\,, \quad H = \tfrac{1}{\log(4/3)} \left(\tfrac{\pi^2}{6} + 2\mathrm{Li}_2(-1/2) - (\log 2) \tfrac{\log 27}{\log 16} \right)$$

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$\Longrightarrow$ proof turns out to be a bit unexpected.

Procedure summarized in

$$(p, q) \mapsto (p', q') = (q - 2^a p, 2^a p)\,,$$

where $a = a(p, q) = \max\{k \geq 0 : 2^k p \leq q\}$.

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Note.

- The map $p/q \mapsto p'/q'$ can be extended to $\mathcal{I} = (0, 1)$

$$T\colon \mathcal{I} \to \mathcal{I}\,, \qquad T(x) = \frac{2^{-a}}{x} - 1\,,$$

  where $a = \lfloor \log_2(1/x) \rfloor$.

- Algorithm gives rise to a continued fraction

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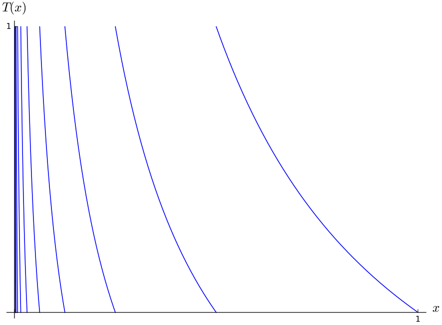
Dynamical system $(\mathcal{I}, T)$



The map $T\colon \mathcal{I} \to \mathcal{I}$

Branches

For $x \in \mathcal{I}_a := [2^{-a-1}, 2^{-a}]$

$$x \mapsto T(x) := \frac{2^{-a}}{x} - 1\,.$$

where $a(x) := \lfloor \log_2(1/x) \rfloor\,.$

Inverse branches

$$h_a(x) := \frac{2^{-a}}{1+x}\,, \quad \mathcal{H} := \{h_a : a \in \mathbb{N}\}\,,$$

and at depth k

$$\mathcal{H}^k := \{h_{a_1} \circ \cdots \circ h_{a_k} : a_1, \ldots, a_k \in \mathbb{N}\}\,.$$

Density transformer

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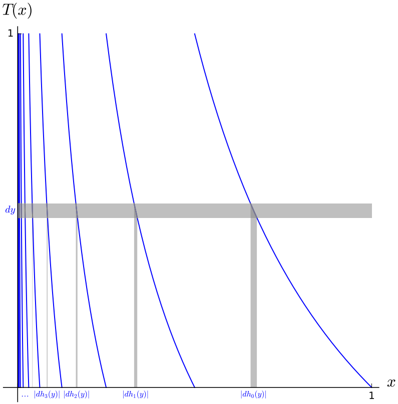

Answer: The density is

$$\mathbf{H}[g](x) = \sum_{h \in \mathcal{H}} \left| h'(x) \right| g\left(h(x)\right)$$
$$= \frac{1}{(1+x)^2} \sum_{a \geq 0} 2^{-a}\, g\left(\frac{2^{-a}}{1+x}\right) .$$

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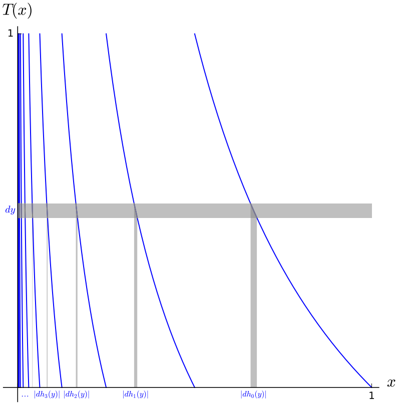

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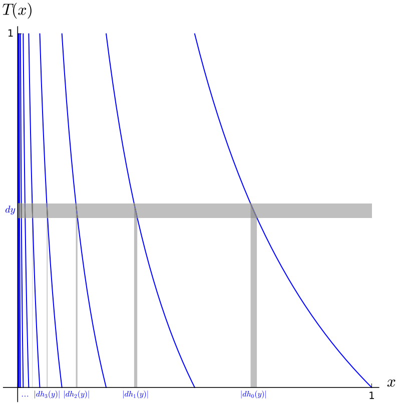

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$$\mathbf{H}^k[g](x) = \sum_{h \in \mathcal{H}^k} \left|h'(x)\right| g\left(h(x)\right).$$

$\implies$ Transfer operator $\mathbf{H}_s$ extends $\mathbf{H}$, introducing a variable s

$$\mathbf{H}_s[g](x) = \sum_{h \in \mathcal{H}} \left|h'(x)\right|^s g\left(h(x)\right).$$

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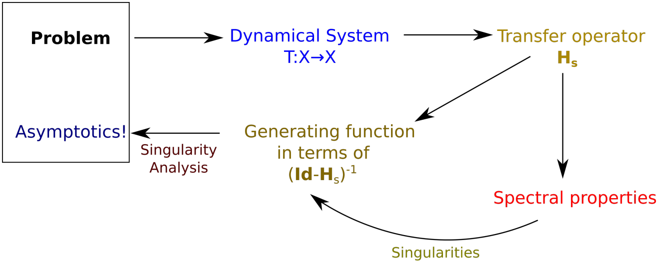

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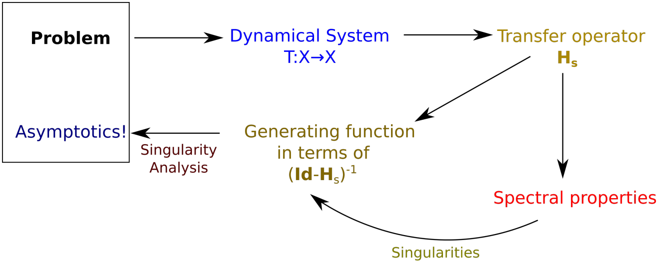

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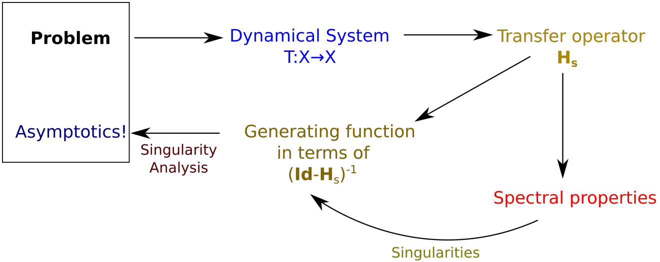

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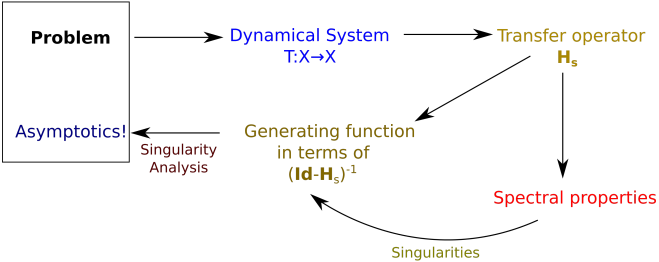

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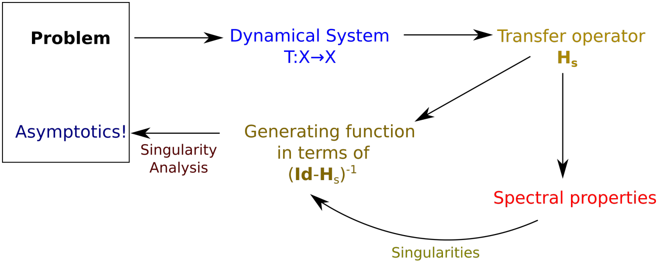

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$\Longrightarrow$ Need a dyadic component to mark these!

Recording the dyadic behaviour

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  $\Rightarrow$ yet the real component that "leads".

Idea works!

Average behaviour of the CL algorithm

Input model:

$$\Omega := \{(p,q) : 0 < p < q,\ \gcd(p,q) = 1\}\,, \quad \Omega_N := \Omega \cap [N] \times [N]\,,$$

take uniform probability on Ω_N

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We have explicit constants

$$E_N[K] \sim \frac{2}{H} \log N, \qquad E_N[S] \sim \frac{\log 3 - \log 2}{2\log 2 - \log 3} E_N[K],$$

here H is known as the entropy of the system,

$$H = \frac{1}{\log(4/3)} \left(\frac{\pi^2}{6} + 2\mathrm{Li}_2\left(-\tfrac{1}{2}\right) - (\log 2)\frac{\log 27}{\log 16} \right),$$

numerically $H \doteq 1.33973\ldots$

The extended dynamical system

⊛ Introduce $\underline{\mathcal{I}} := \mathcal{I} \times \mathbb{Q}_2$ and $\underline{T} \colon \underline{\mathcal{I}} \to \underline{\mathcal{I}}$ as follows

$$\underline{T}(x, y) = \left(T_a(x), T_a(y)\right),$$

for $x \in \mathcal{I}_a = [2^{-a-1}, 2^{-a}]$. This gives inverse branches

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$\Longrightarrow$ Consider related measure $\tilde{\nu}$ on $\mathbb{Q}_2$!

⇒ *extended* transfer operator $\underline{\mathbf{H}}_s$.

Functional space $\mathcal{F}$ for the extended operator $\underline{\mathbf{H}}_s$

Real component directs the dynamical system:

- *sections* F_y fixing $y \in \mathbb{Q}_2$ asked to be $C^1(\mathcal{I})$.
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We can finish the dynamical analysis!

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4. **Expansion for real numbers:** work in progress!