On self-assembly of aperiodic tilings

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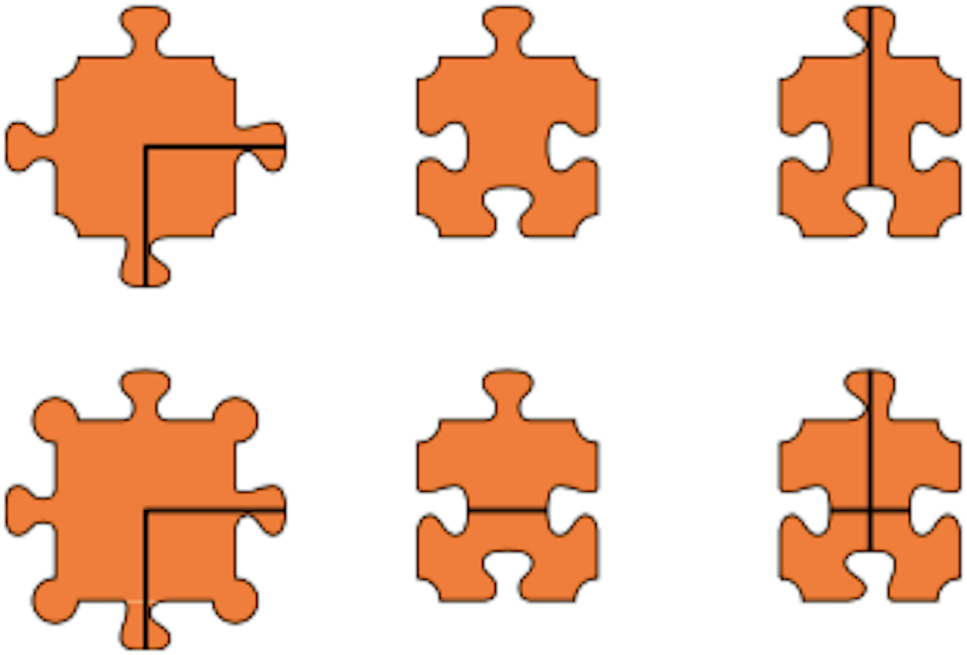
Tiling

By a tiling of $\mathbb{R}^n$ we mean a representation of $\mathbb{R}^n$ as a union of "tiles" where:

- there is a fixed finite set $S = \{p_1, p_2, \ldots, p_l\}$ of "*prototiles*," which are pairwise homeomorphic to the closed ball;
- each tile is an isometric copy of some prototile;
- the interiors of the tiles do not overlap.

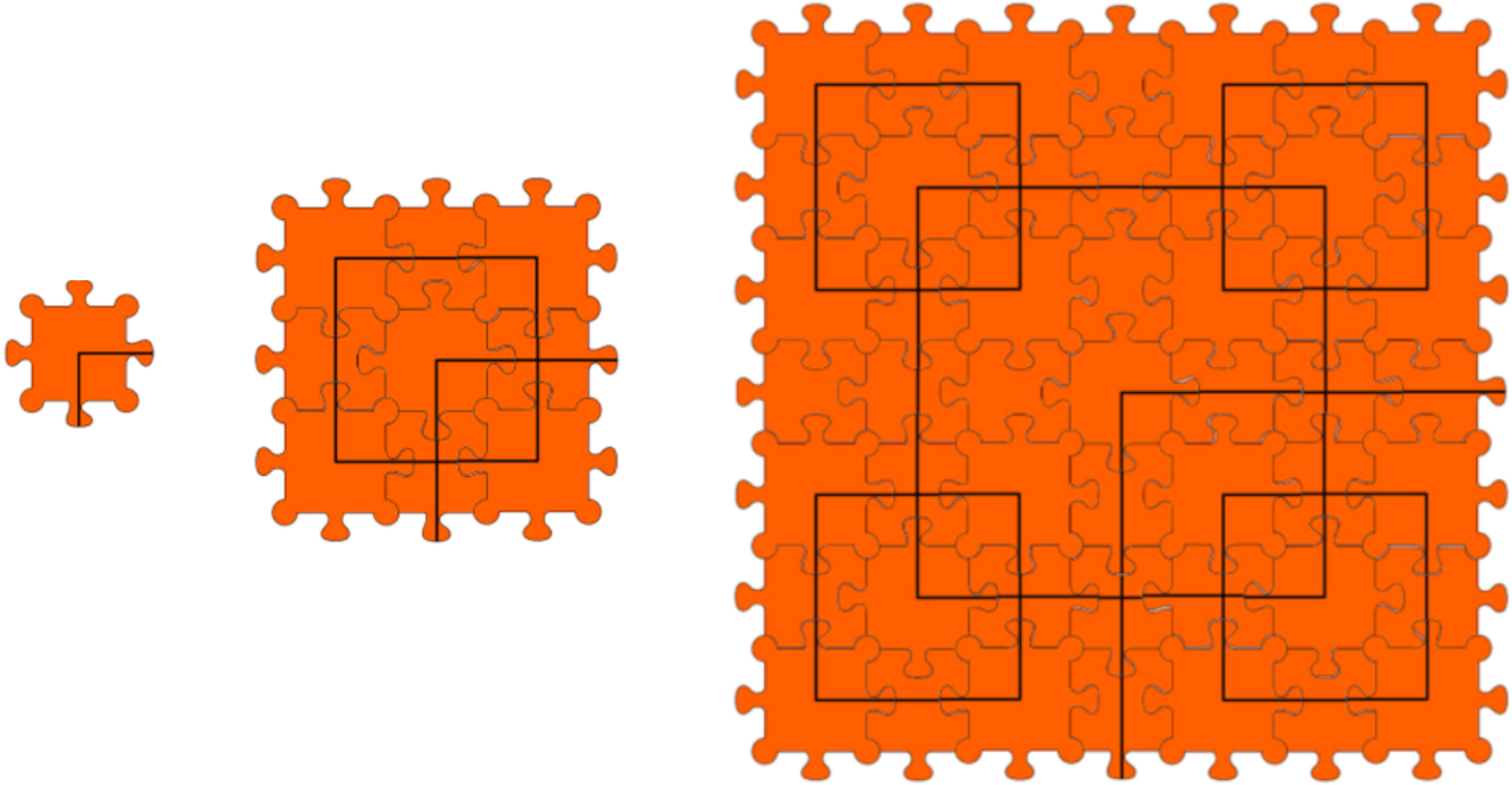
Aperiodic tilings

- A set of tiles is called *aperiodic* if copies of them can cover the whole plane (*the tiling*) but only in a non-periodic way.

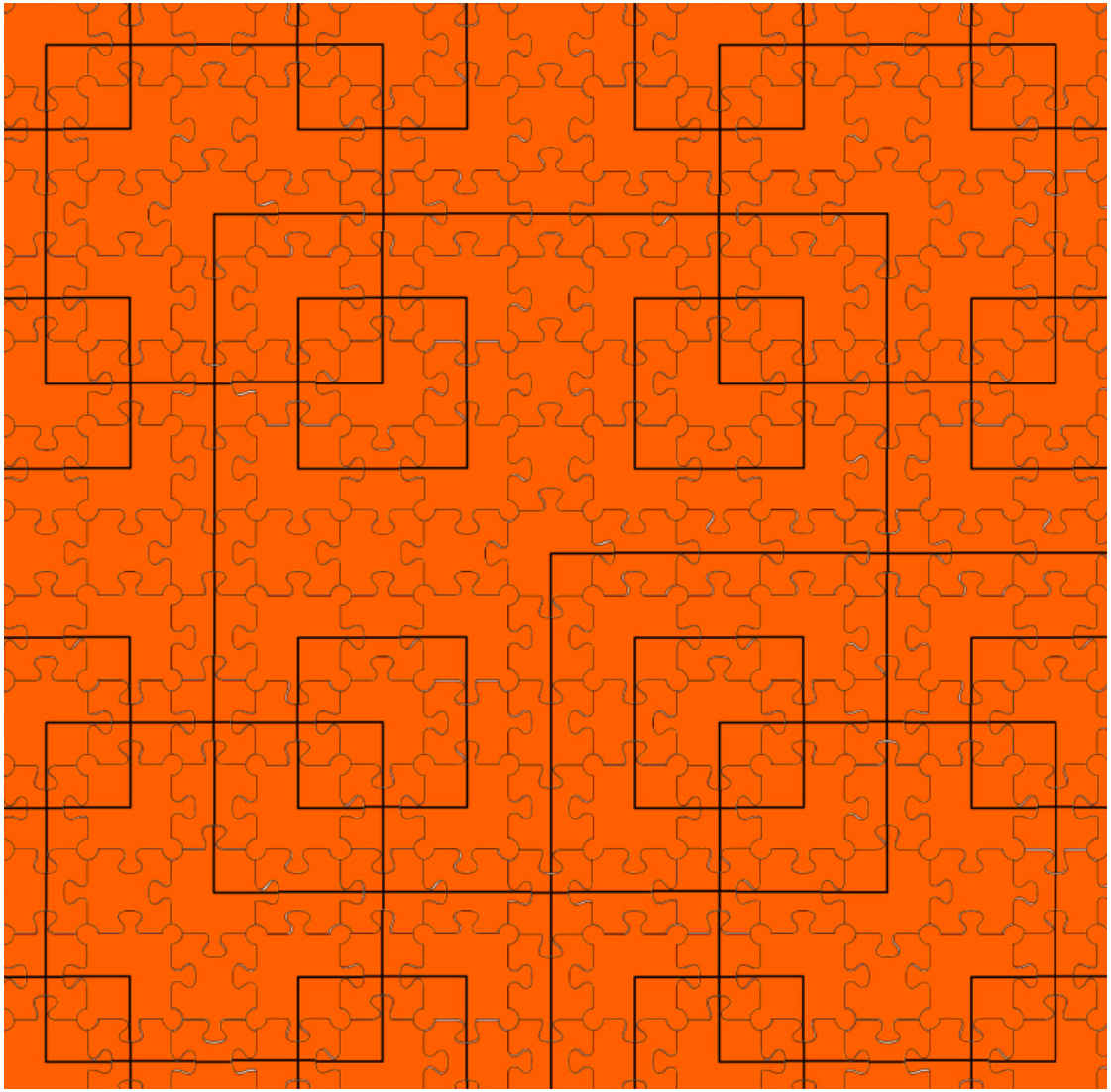


Prototiles of Robinson tiling

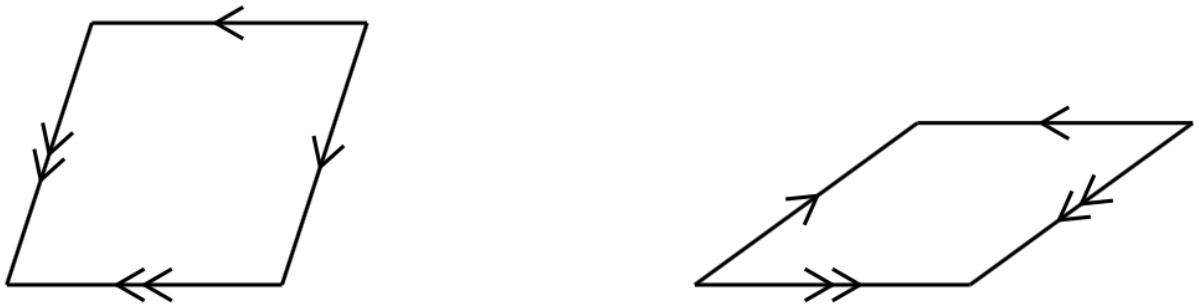
Example: Robinson tiling



Example: Robinson tiling



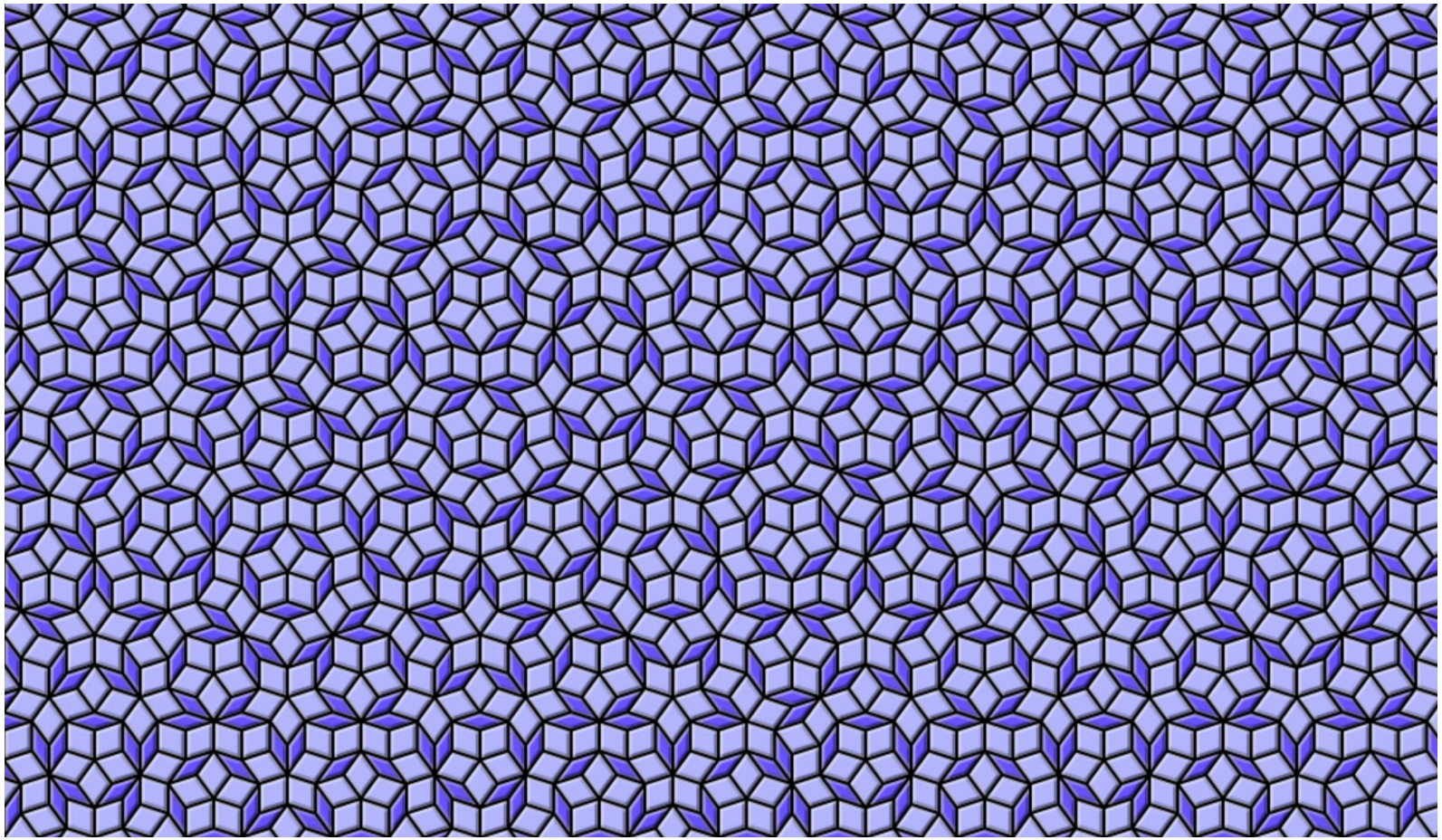
The most famous example: Penrose tiling



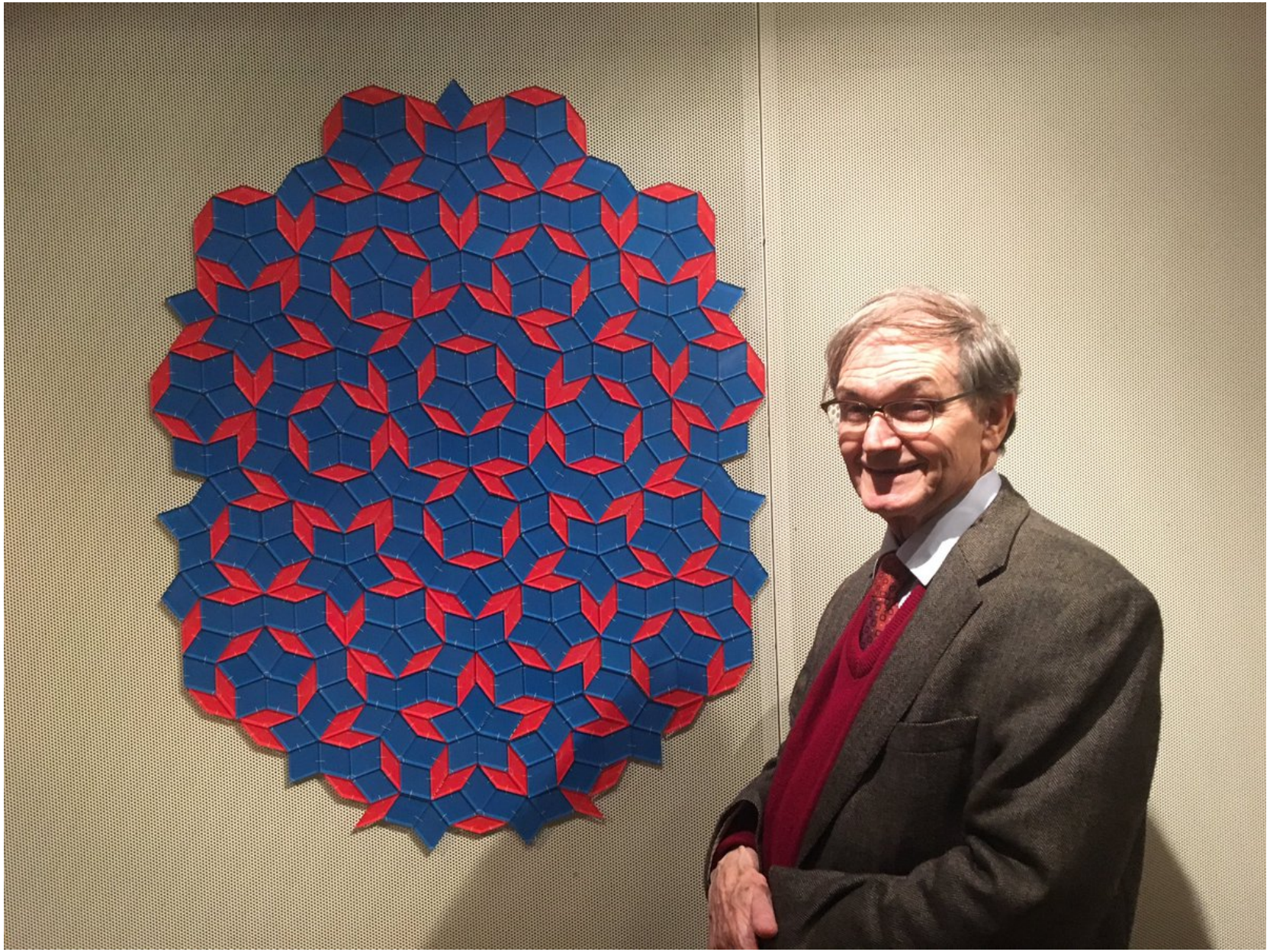
Matching rules for Penrose rhombuses.

- The thin rhomb has four corners with angles of 36, 144, 36, and 144 degrees;
- The fat rhomb has angles of 72, 108, 72, and 108 degrees.

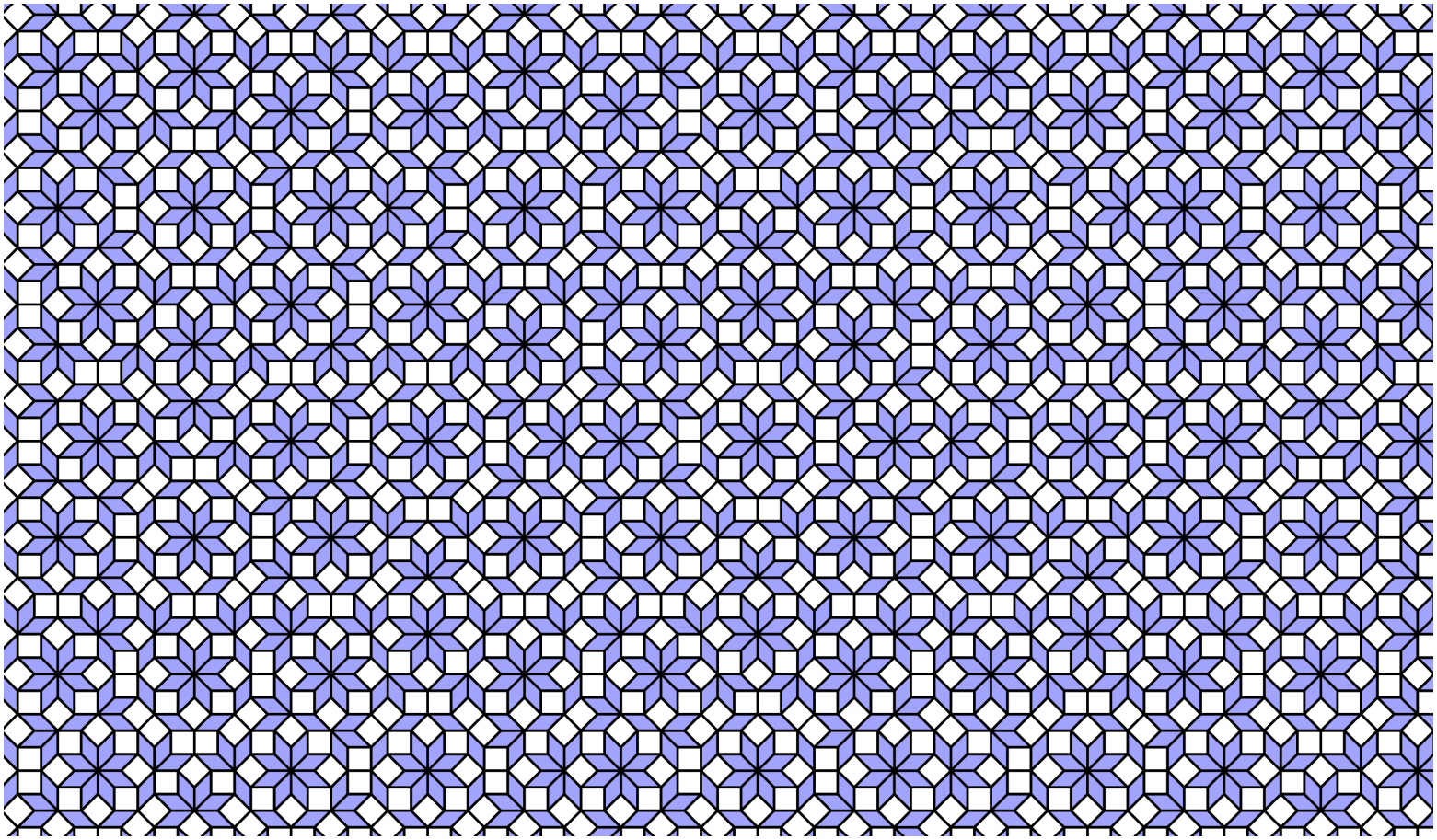
The most famous example: Penrose tiling



The most famous example: Penrose tiling



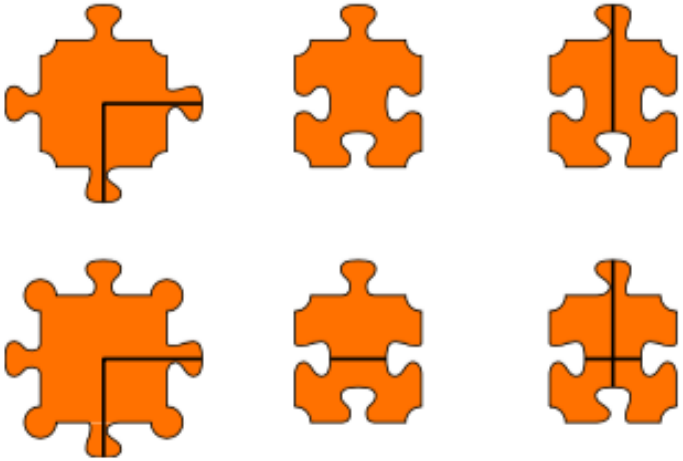
Example: Ammann-Beenker

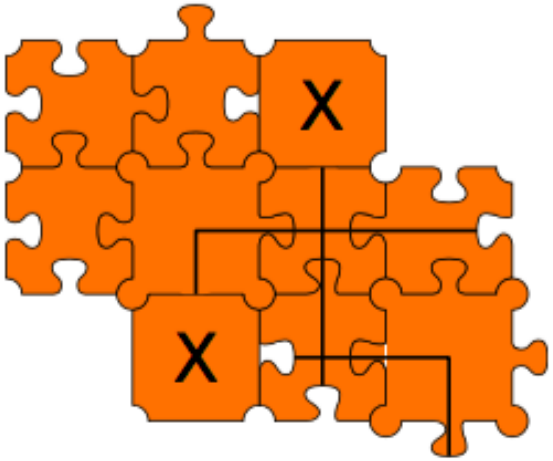


Motivation

- Rapid development of aperiodic tilings started after discovery of quasicrystals in 1982 by Dan Shechtman (who won the Nobel prize in chemistry in 2011)
- The atomic arrangement of a quasicrystal breaks the periodicity (meaning it has no translational symmetry)
- Due to specific local structure of these materials the growth process of such crystals is still poorly understood.

Self-assembly and deceptions





- The danger is that adding tiles one by one may lead to a pattern that cannot be further extended.

Deceptions

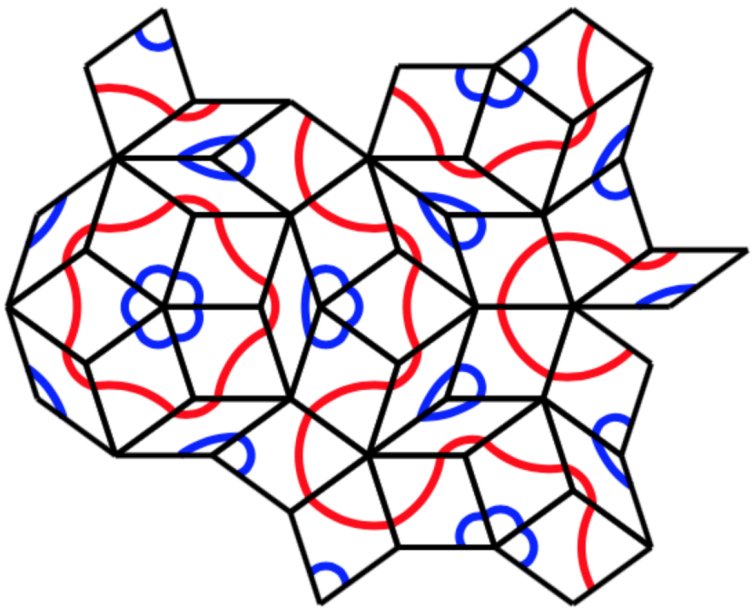
Definition

A patch will be *regular* if it is homeomorphic to the closed ball, and is of *order* r if it covers some disc of radius r.

A regular patch P will be a *deception* of order r if:

- every connected subpatch of P of cardinality less than r is a subset of some tiling of the plane, and
- P is not a subset of any tiling of the plane.

Example



Deception for Penrose tiling.

Deceptions

Theorem (Dworkin-Shieh, 1993)

Any aperiodic protoset admits deceptions of all orders.

Question

Is it possible to define a *local growth process* that never produces a deception?

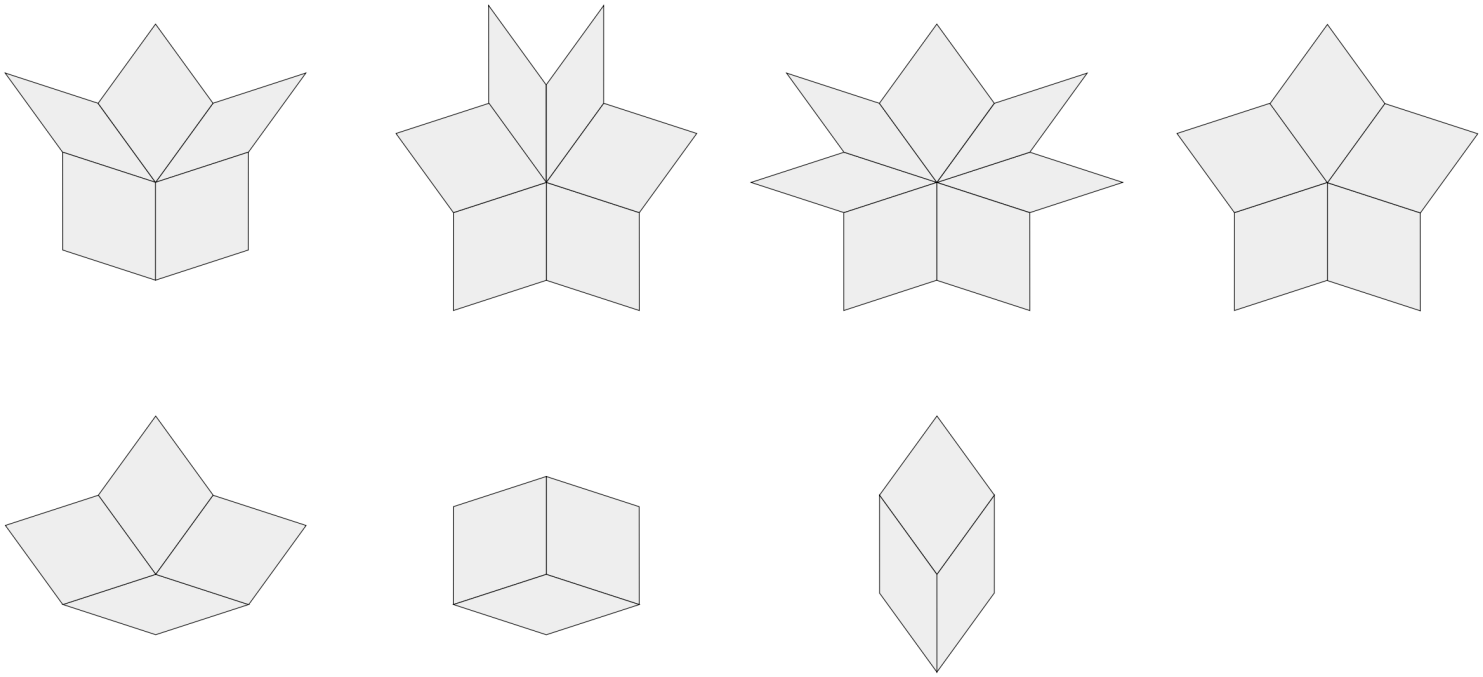
The meaning of local growth is:

- tiles must be added one by one;
- decision on what tile to add must be done only by inspecting local configuration around the chosen edge (to do nothing is a valid decision);
- the edge must be chosen at random (optionally).

Vertex-atlas

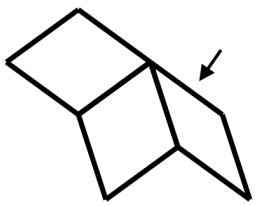
Definition

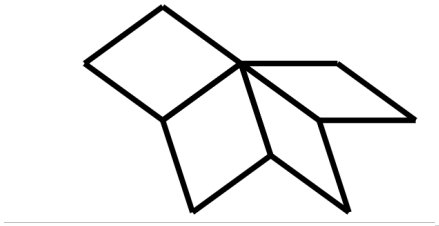
Given a tiling $\mathcal{T}$ and a real number $r > 0$ the set of all patches centered at some vertex $x \in \mathcal{T}$ of size r will be called an *r-atlas* of $\mathcal{T}$ and denoted as $\mathcal{A}(r)$.



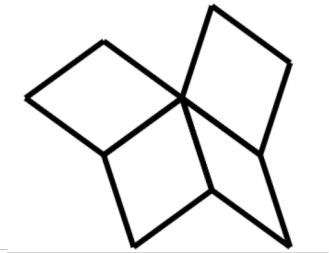
Forced tiles

The idea is to add tiles that are forced by some vertex atlas $\mathcal{A}(r)$ (with r big enough). Example:





(a)

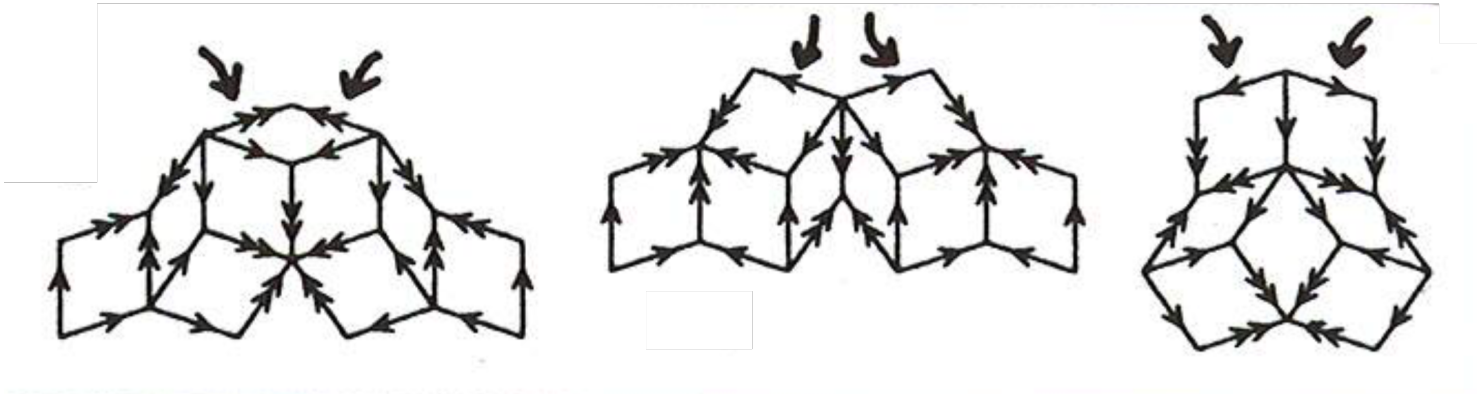


(b)

- (a) is consistent with vertex-atlas;
- (b) is not.

Edge types

- *Forced* – only one tile fits (with respect to vertex-atlas);
- *Unforced* – there is a choice;
- *Special*:



Self-assembly algorithm (Socolar, 1991)

- Start with a finite patch of Penrose tiling;
- Keep adding the forced tiles one by one until it is possible;
- When there are none forced tiles left add the fat tile to randomly chosen special site;
- Repeat.

Theorem (Socolar, 1991)

The algorithm can build any tiling form Penrose family of tilings.

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Theorem (Socolar, 1991)

The algorithm can build any tiling form Penrose family of tilings.

- However, this algorithm is *not* local.
- The algorithm modified to work with Ammann-Beenker tilings familly will not work!

Demonstration

Demonstration: Penrose.

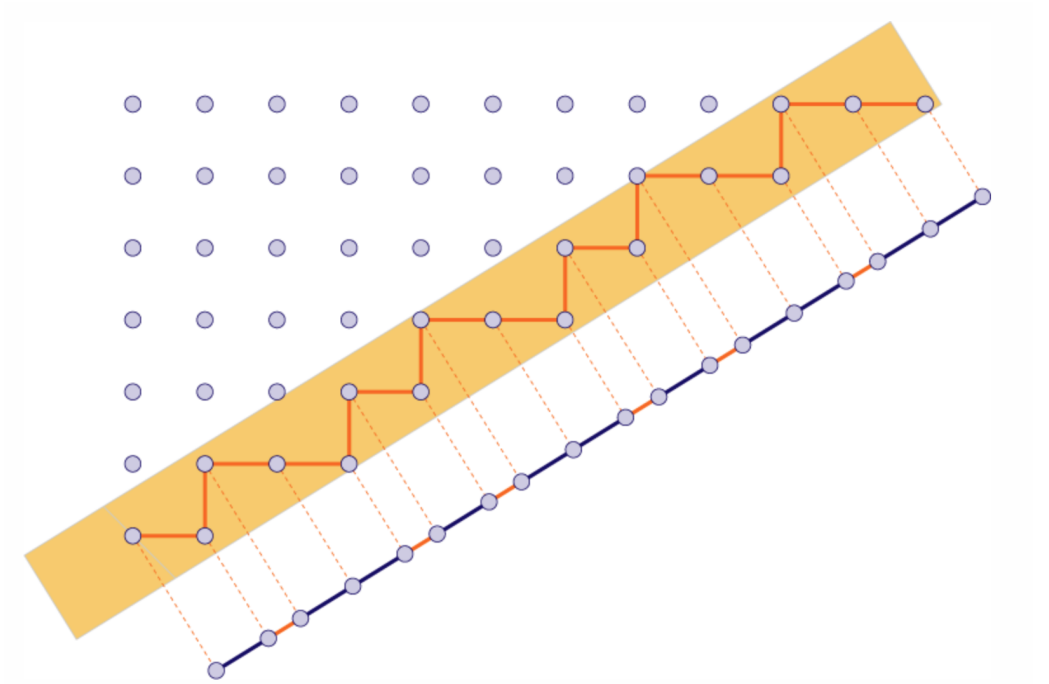
Canonical cut-and-project

Definition (Planar tiling)

Let E be a d-dimensional affine space in $\mathbb{R}^n$ such that $E \cap \mathbb{Z}^n = \varnothing$. Select the d-dimensional faces with vertices in $\mathbb{Z}^n$ lying in the *strip* $S = E + [0,1]^n$. Project them onto E to get a so-called *planar* $n \to d$ tiling. E is called the *slope* of a tiling.

Classical example: Fibonacci tiling

In the standard lattice $\mathbb{Z}^2$, choose a stripe with slope $\frac{1}{\tau}$ (where tau is the golden ratio). Take all lattice points within the strip and project them orthogonally to a line parallel to the strip. Interpreting the point set as a partition of the line into long and short intervals yields a tiling of the line.



Canonical cut-and-project

The family of Penrose tilings can be defined both through the set of prototiles with matching rules and through the cut-and-project scheme!

Theorem (De Bruijn, 1981)

Penrose tiling is planar $5 \to 2$ *tiling with the slope* E *generated by*

$$u = \begin{pmatrix} 1 \\ \cos(2\pi/5) \\ \cos(4\pi/5) \\ \cos(6\pi/5) \\ \cos(8\pi/5) \end{pmatrix} \qquad v = \begin{pmatrix} 0 \\ \sin(2\pi/5) \\ \sin(4\pi/5) \\ \sin(6\pi/5) \\ \sin(8\pi/5) \end{pmatrix}$$

Necessary conditions

Definition (Local rules)

A d-plane $E \subset \mathbb{R}^n$ is said to have local rules if there is a finite set of patterns so that any $n \to d$ tiling without any of these patterns is planar with the slope parallel to E.

The family of Penrose tilings do have local rules and of Ammann-Beenker do not have them!

Conjecture

In order to have a local self-assembly algorithm for a planar tiling it is necessary for the slope of the tiling to admit a local rules.

Is it sufficient?

Local algorithm

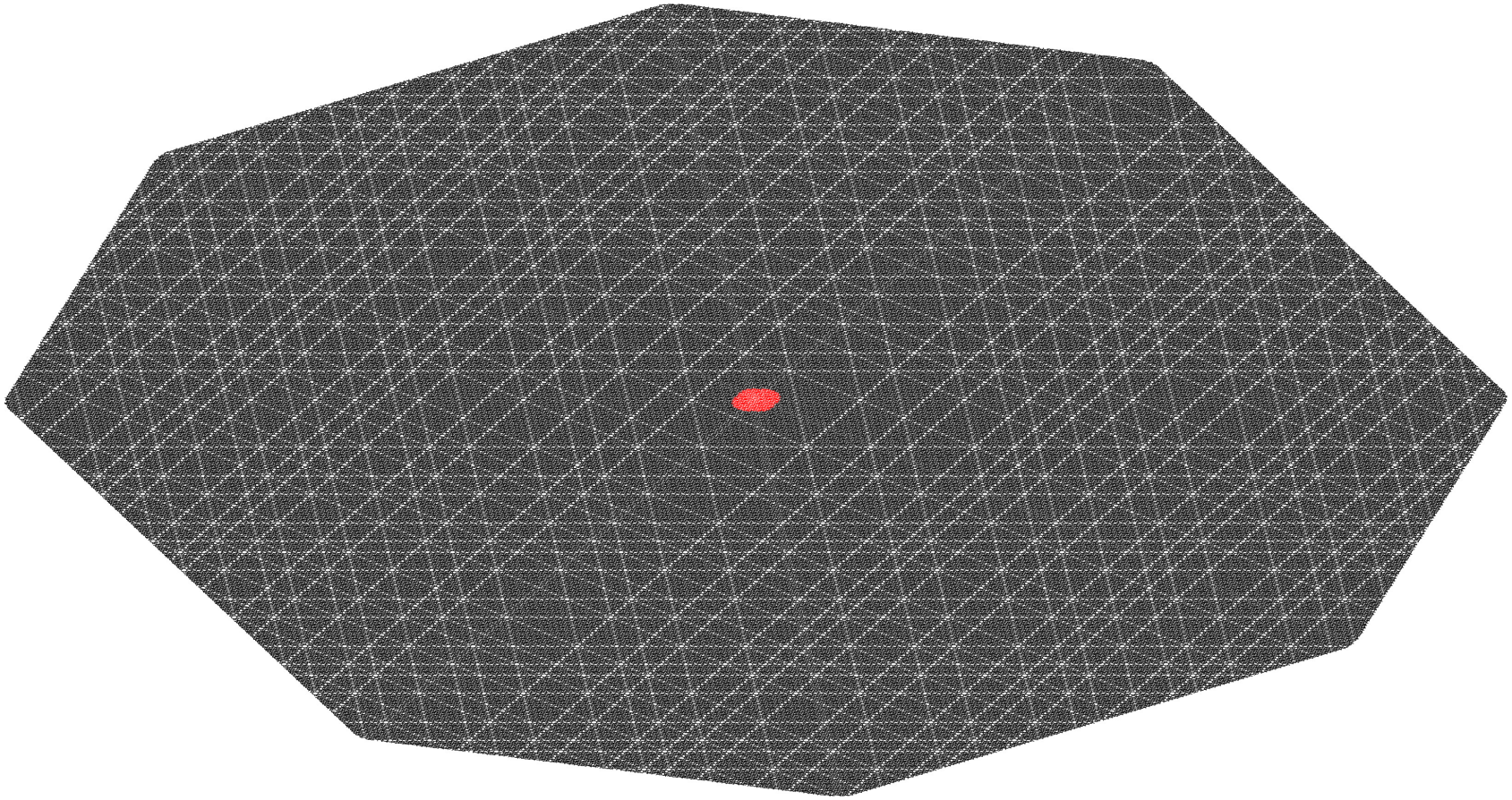
Choose your favourite family of planar tilings with local rules. Given an integer $r > 0$, the atlas $\mathcal{A}(r)$ and a finite patch S from the chosen family, we define an local growth algorithm:

- Pick uniformly at random an open vertex v of S and let $P(v, r)$ be the patch of size r and center $v \in S$;
- Consider the set F of all the elements $a \in$ the vertex-atlas $\mathcal{A}$ that 'match' the local vertex configuration $P(v, r)$;
- Then add to S all the vertices from the set F which *agree* with each other (the forced vertices):
- Repeat.

Demonstration

Demonstration: Golden-Octagonal tiling.

Golden-Octagonal tiling

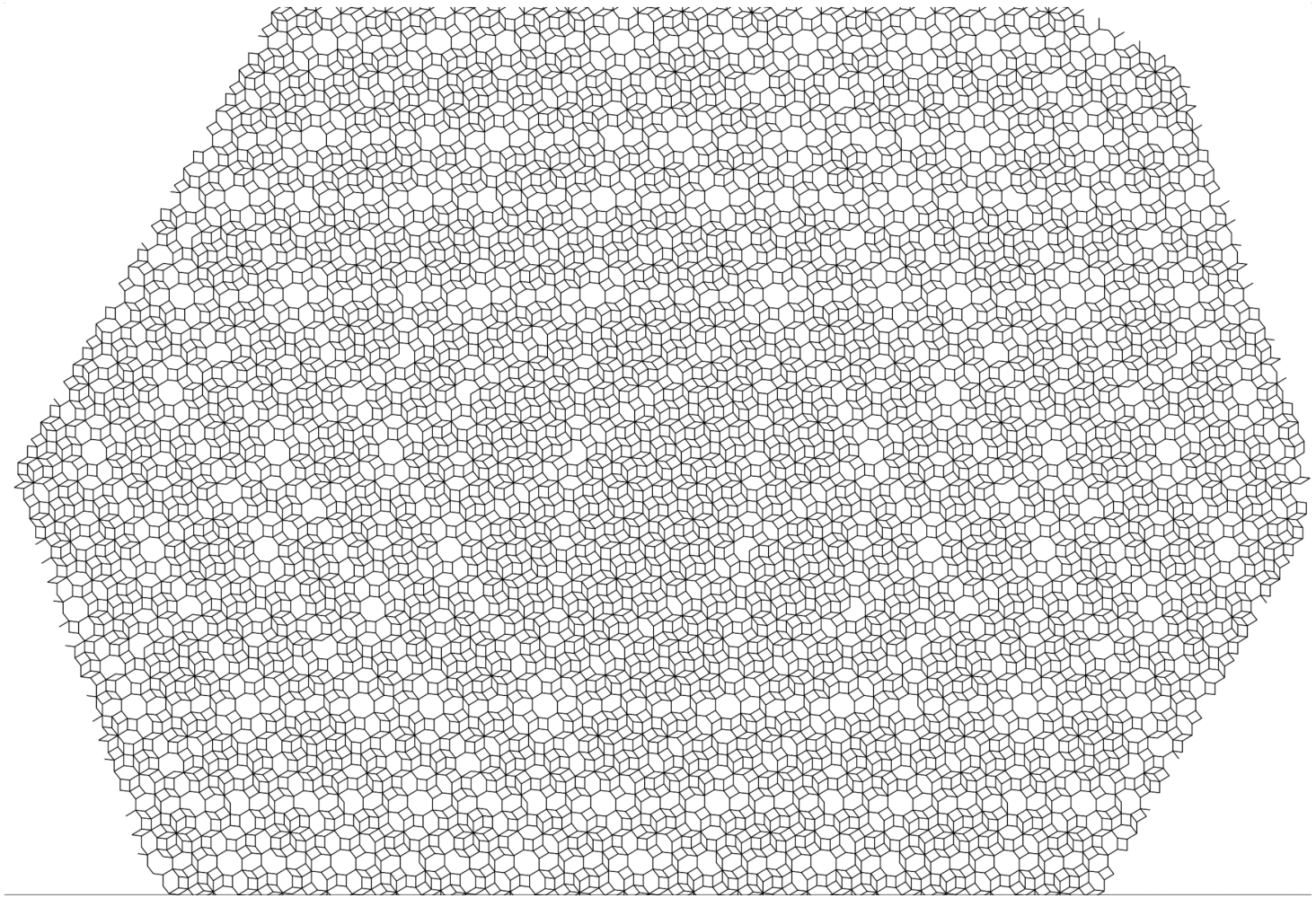


Main conjecture

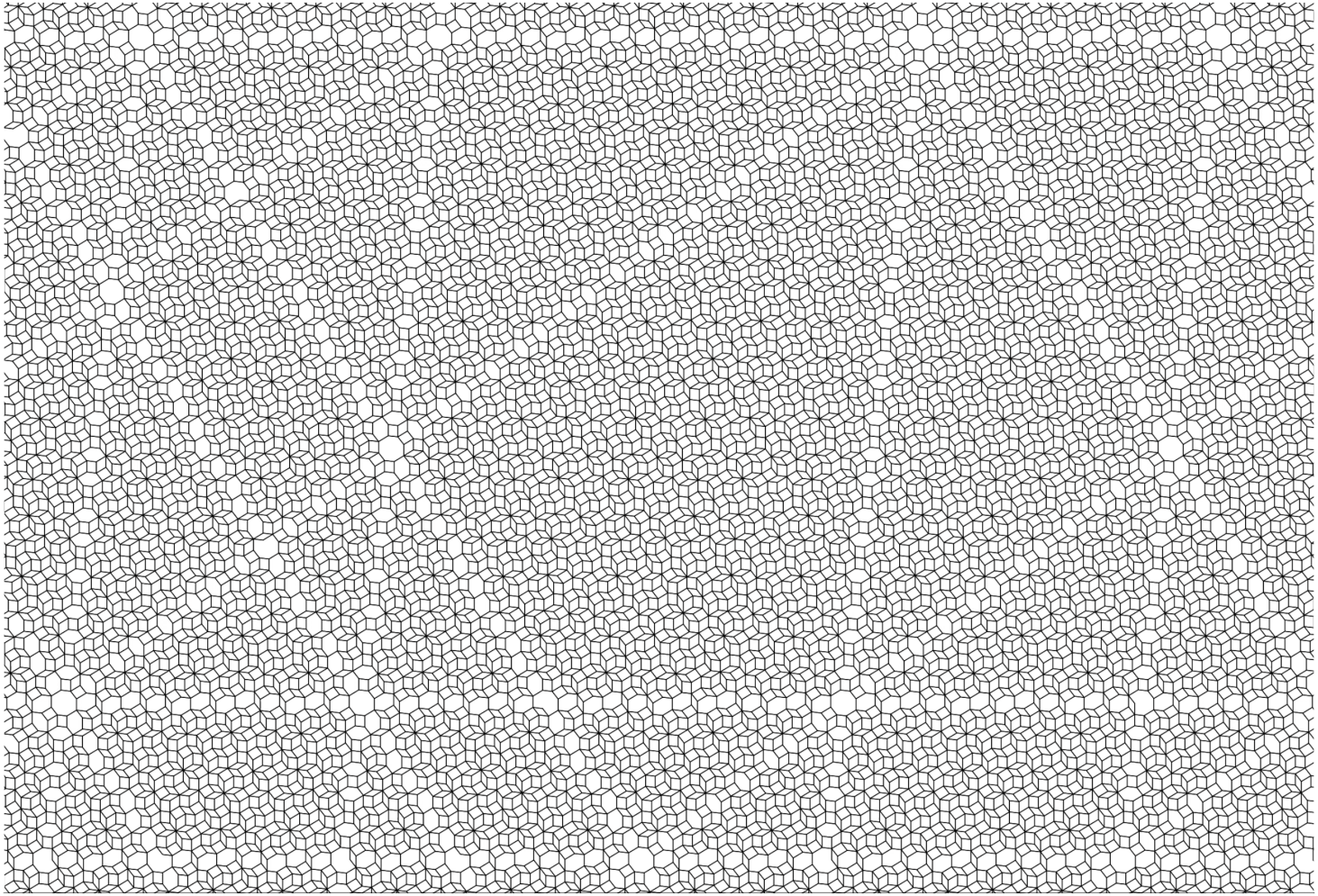
Conjecture

Consider a family of cut-and-project tilings with local rules. For any $\varepsilon > 0$ one can choose a seed, such that above local self-assembly algorithm will produce a tiling of $\mathbb{R}^n$ excluding measure ε.

Smaller seed



Bigger seed

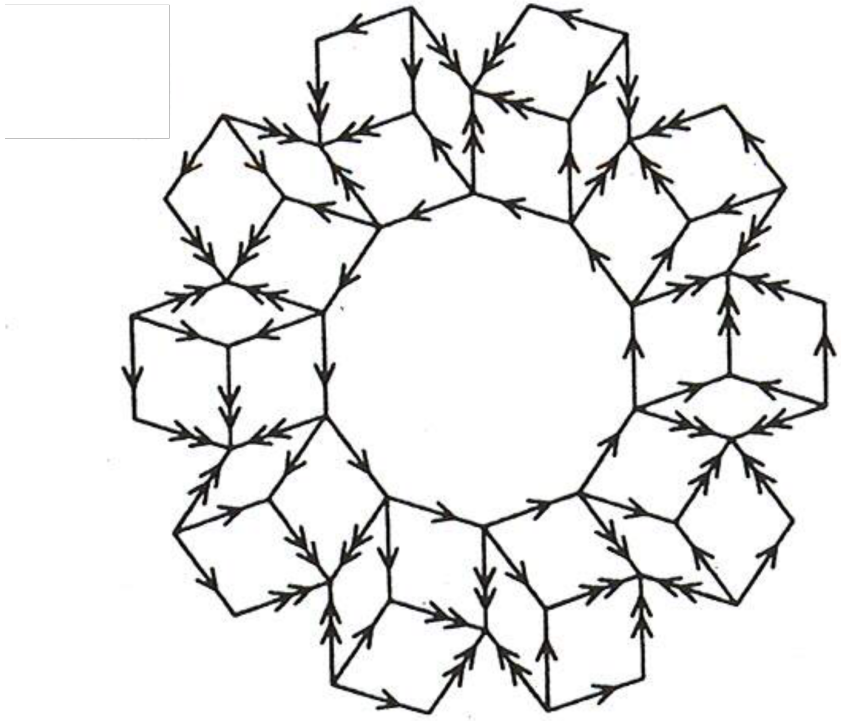


Defective seeds

This algorithm is indeed local but the downside is it will not cover the whole plane, there will be infinitely many empty spots. Is it possible to overcome this issue? The trick is to use defective seeds!

Defective seeds

By using defective patch as a starting seed one could grow a tiling of a plane except for a finite region! Why some defective seeds work?



Example of a such seed for Penrose tiling.

Demonstration

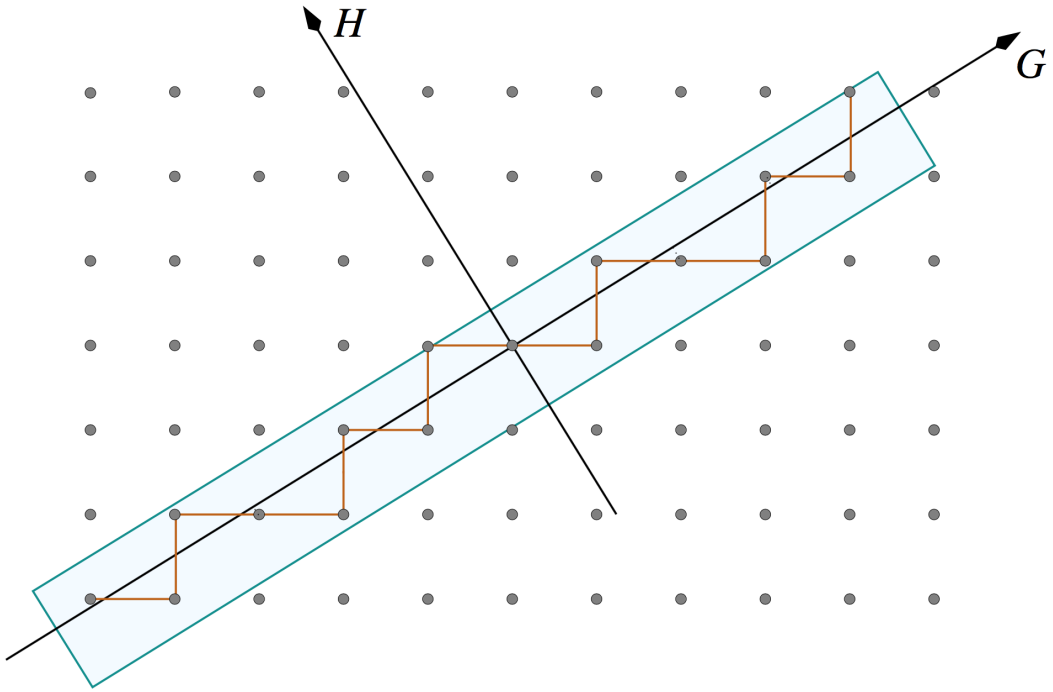
Demonstration.

Window

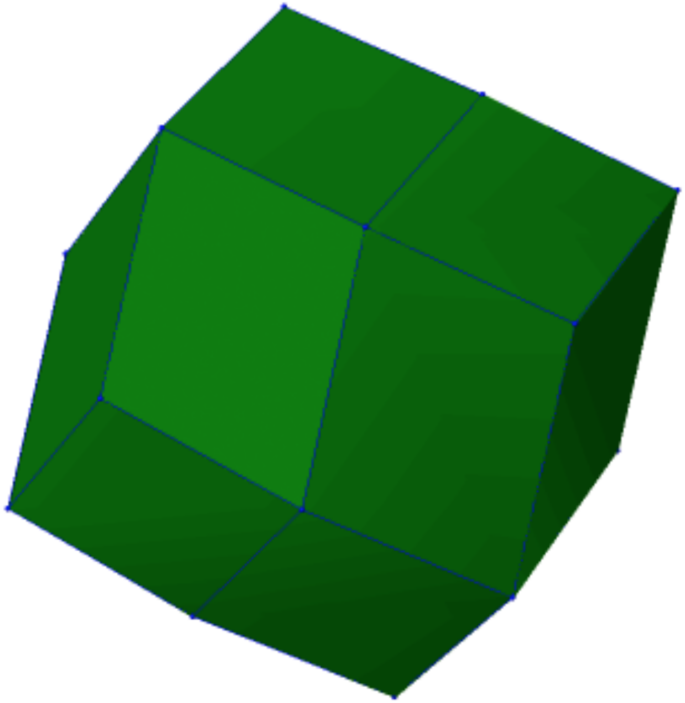
Definition (Window)

The *window* W of a planar tiling with a slope $E \subset \mathbb{R}^n$ is the orthogonal projection of $[0,1]^n$ onto $E^\perp$, where $E^\perp$ is a complementary space to E:

$$W = \pi^\perp([0,1]^n).$$



Window



The window for Penrose tilings.

Window

To every pattern P we can assign a region in the window:

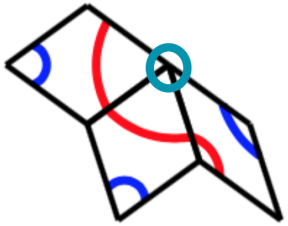
$$R(P) = \bigcap_{x:\pi(x)\in V(P)} (W - \pi^{\perp}(x)).$$

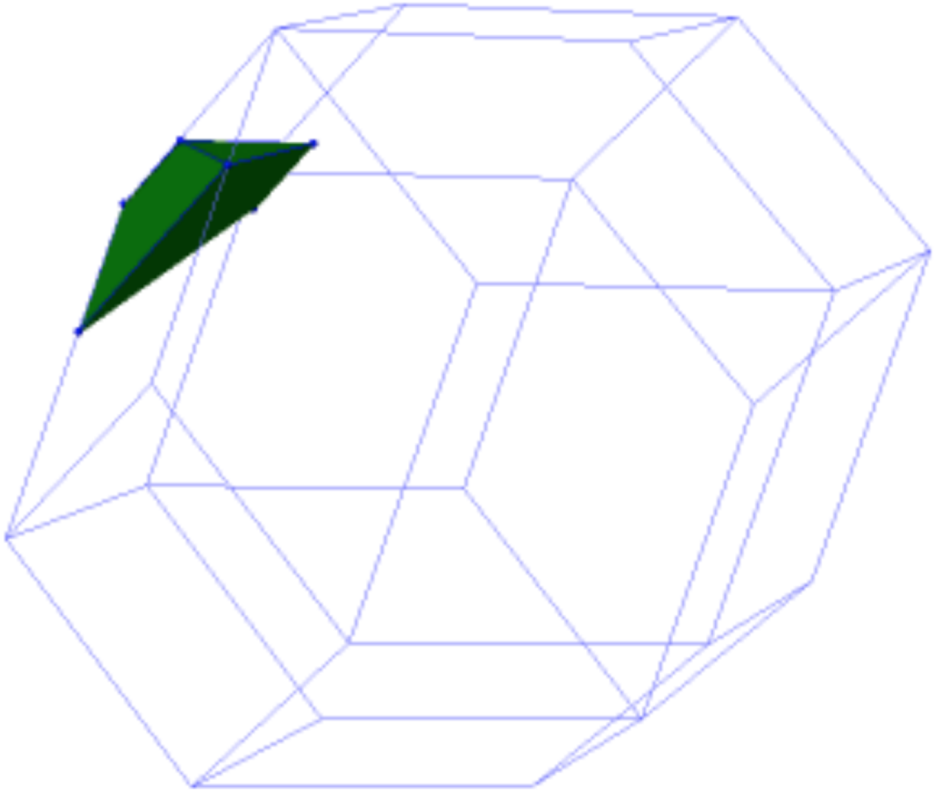
Proposition

In order for pattern P to appear in a tiling it is necessary that

$$R(P) \neq \varnothing.$$

Examples





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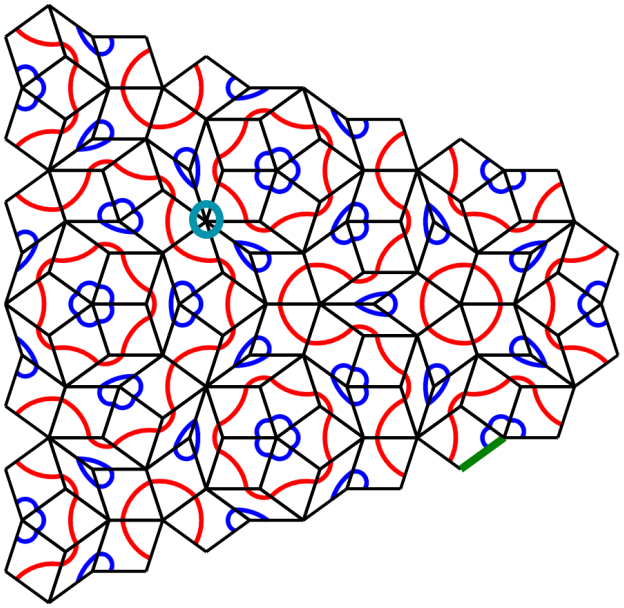
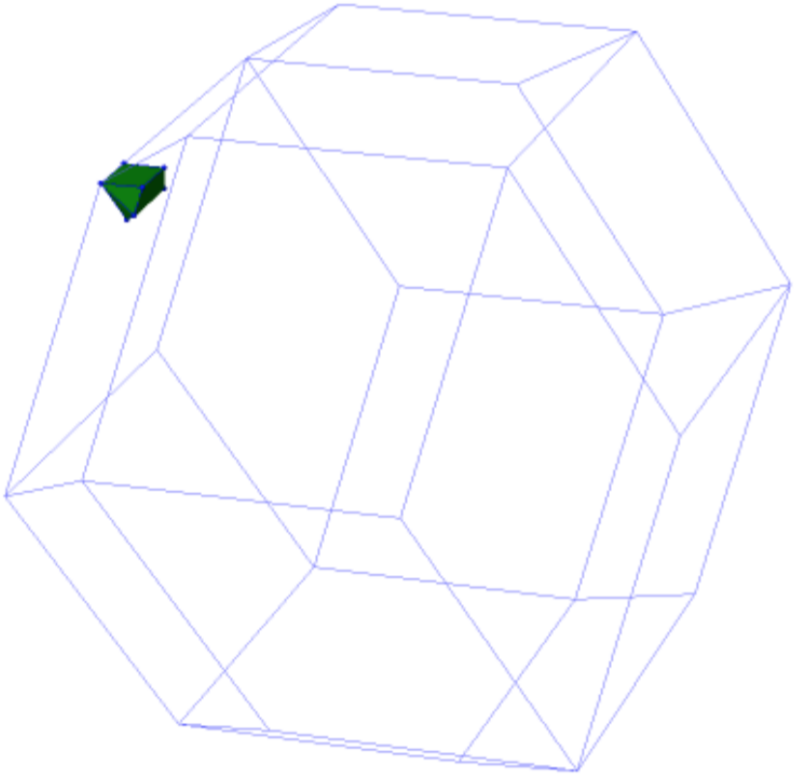
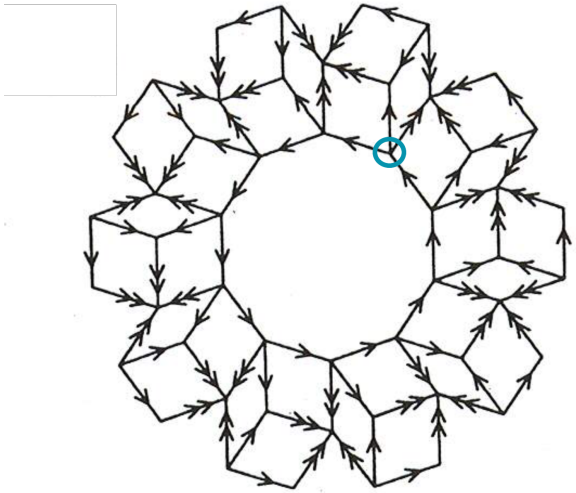


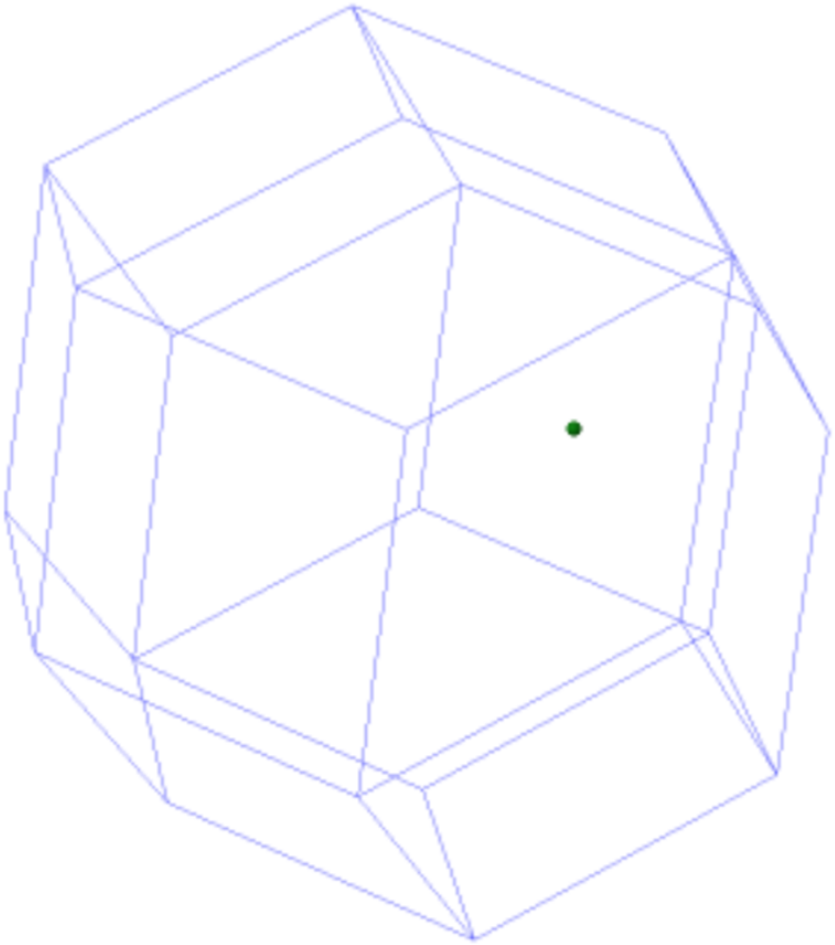


Examples

$$R(Tiling) = \{point\}.$$

Examples





Conjecture

For all the planar tilings with local rules there is a set if defective seeds so that starting the growth with such seeds will give you a tiling of the whole plane except for a finite region.

Thank you for your attention!

Examples