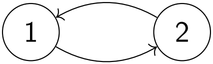
Complete Simulation of Automata Networks

Florian Bridoux
Collaborators: Alonso Castillo-Ramirez, and Maximilien Gadouleau

Automata networks (ANs) : update schedule



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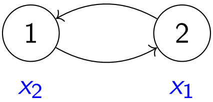

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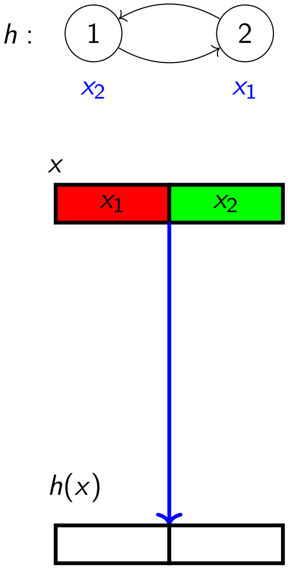

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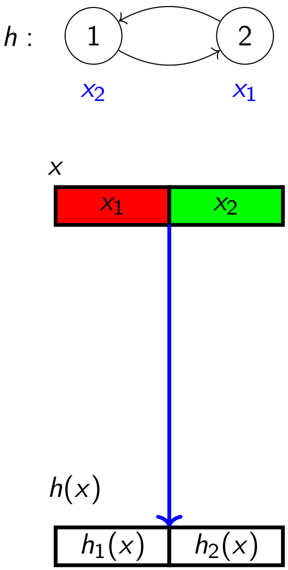

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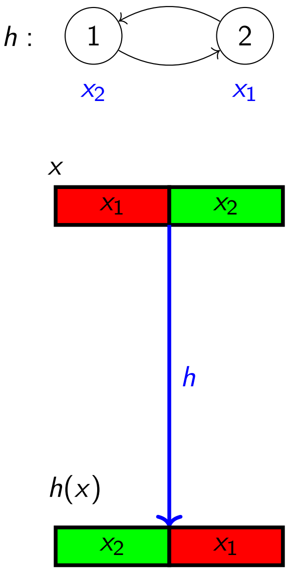

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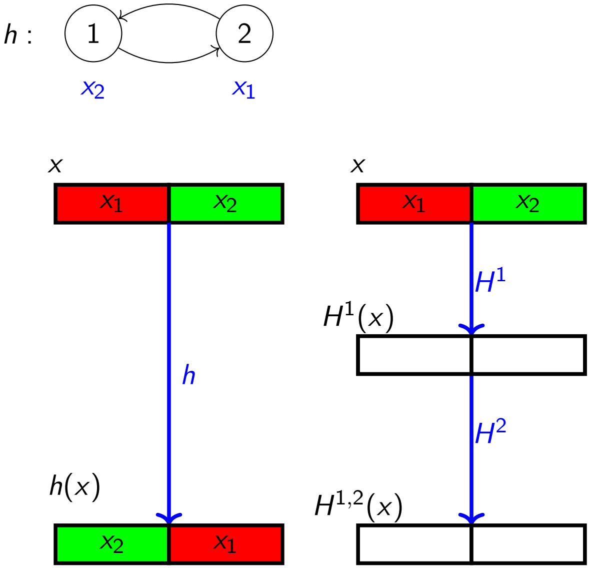

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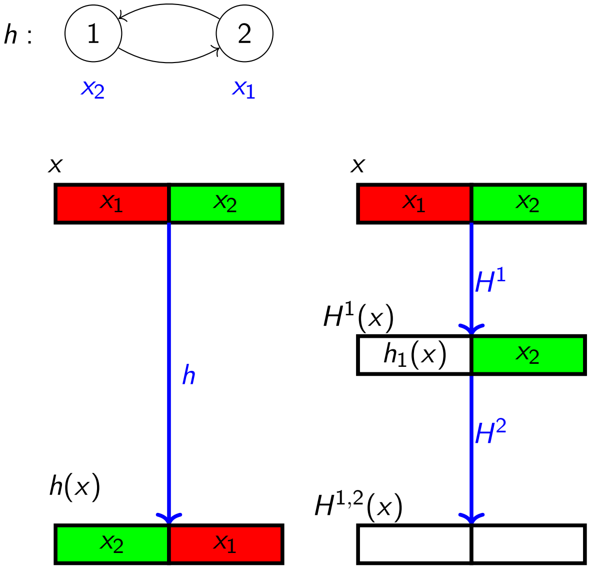

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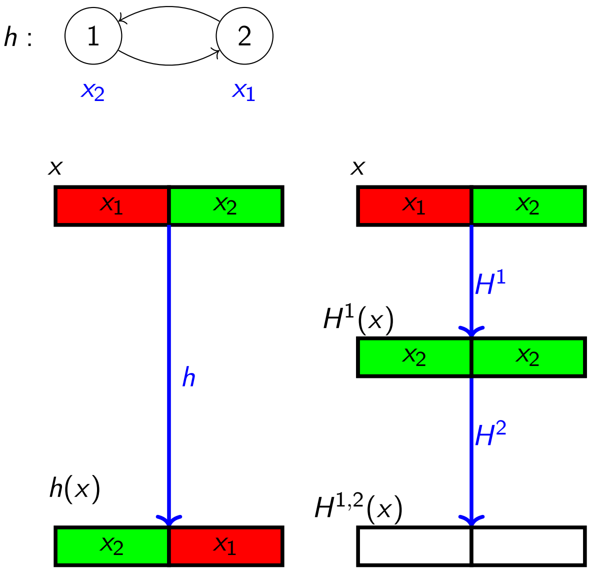

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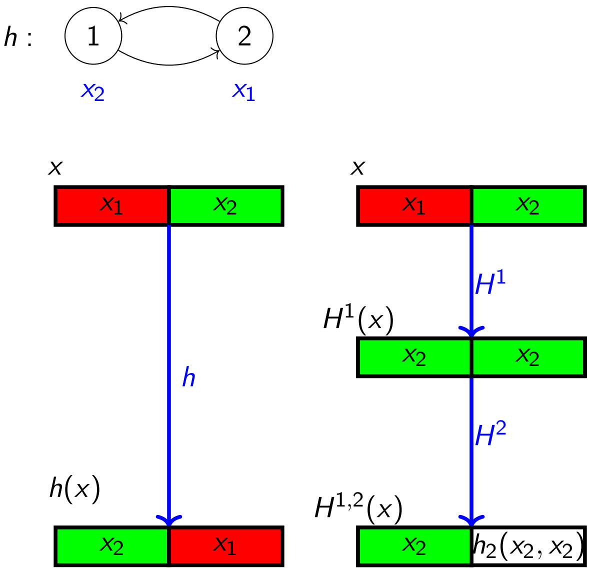

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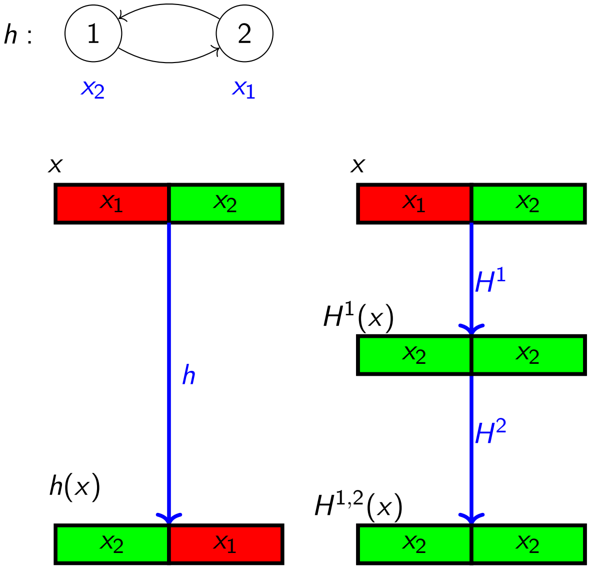

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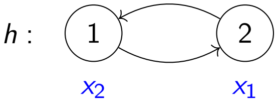

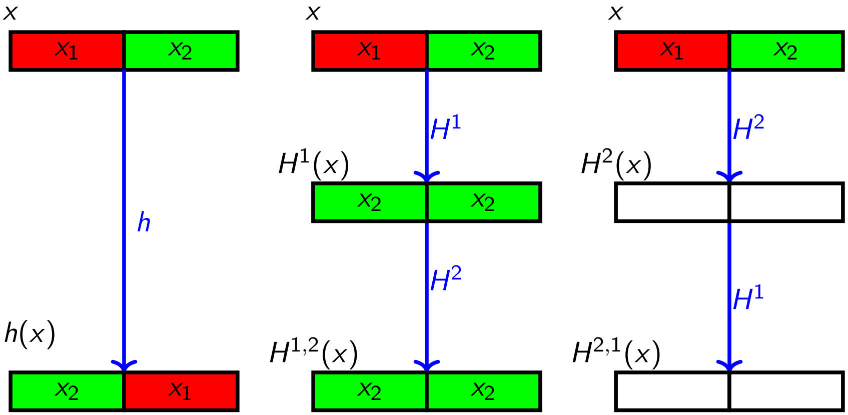

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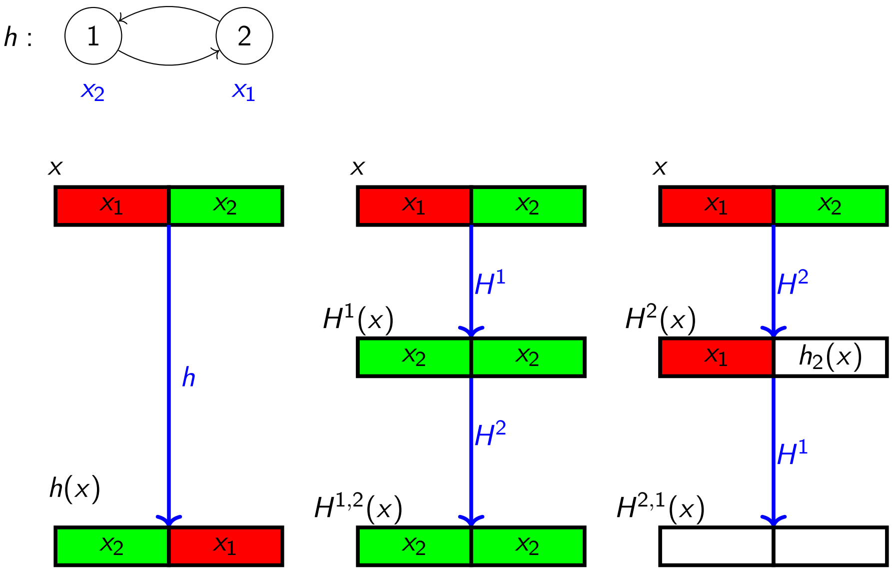

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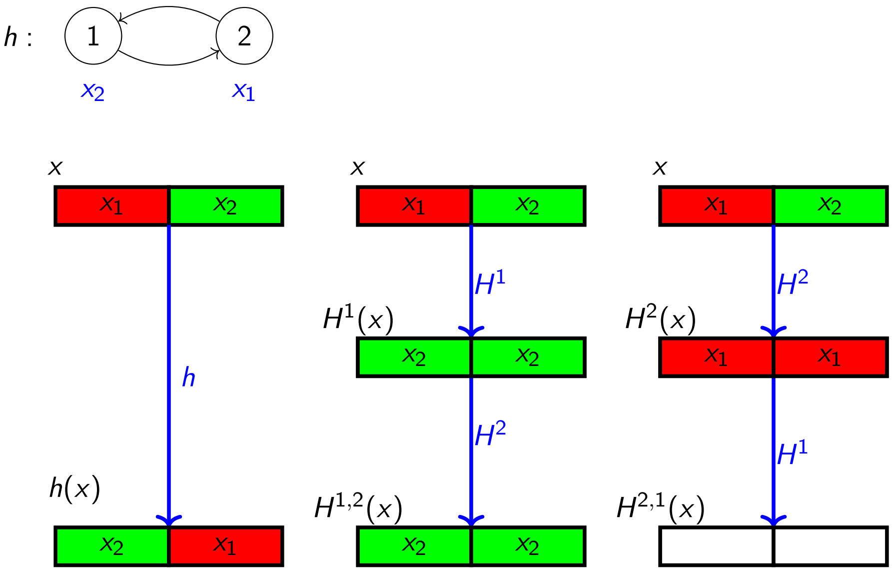

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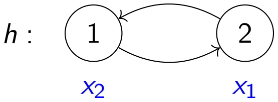

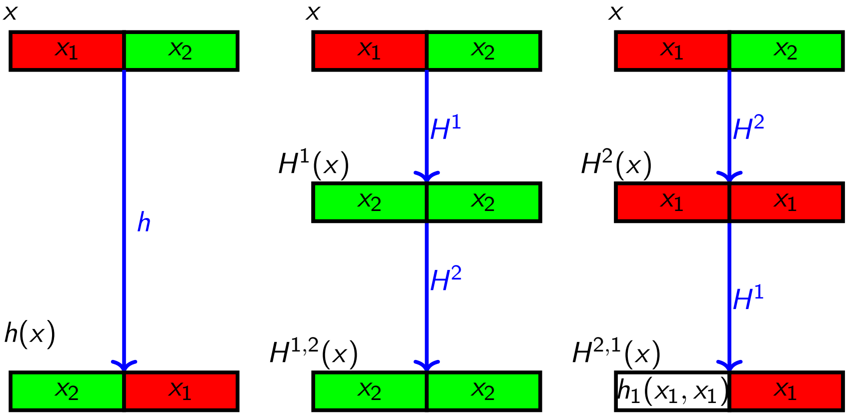

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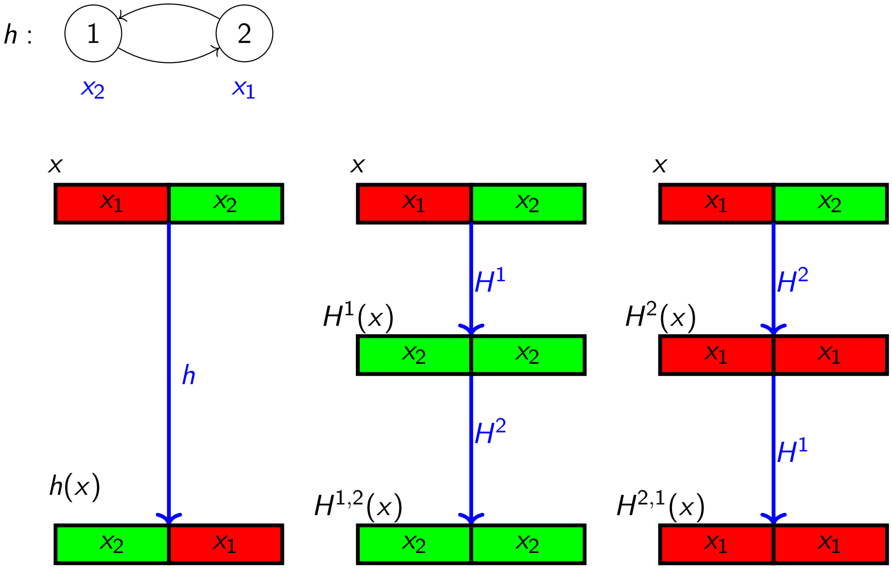

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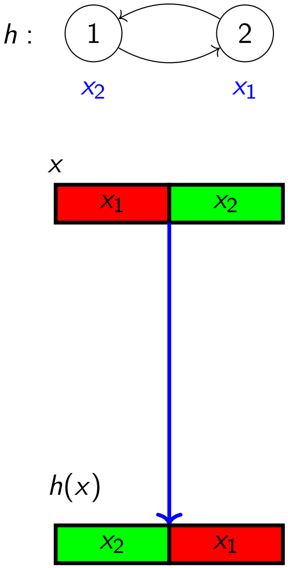

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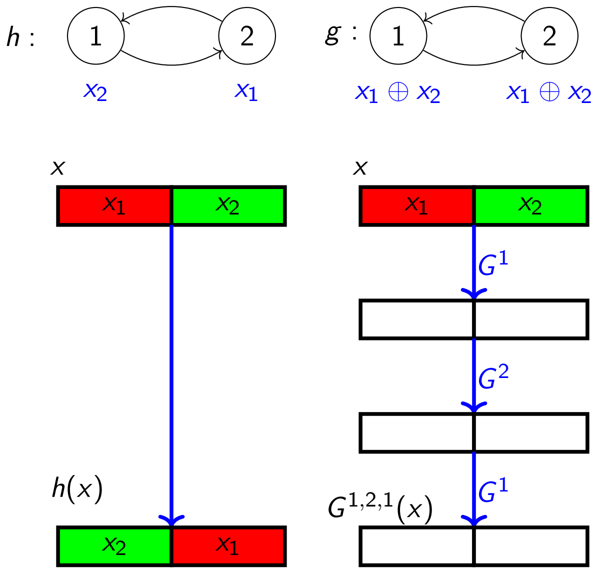

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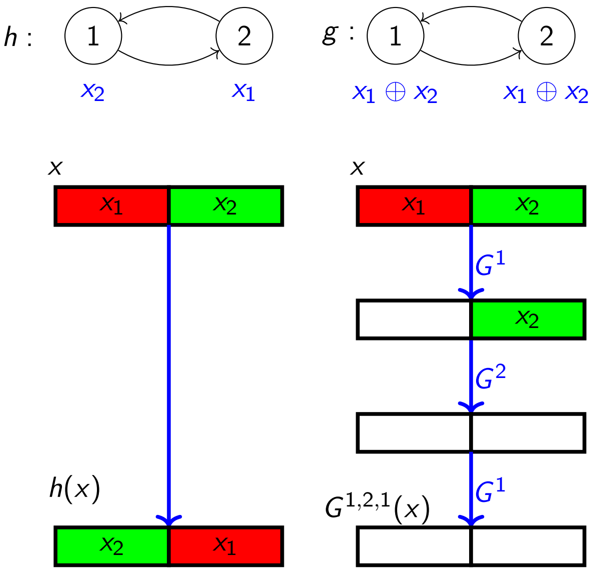

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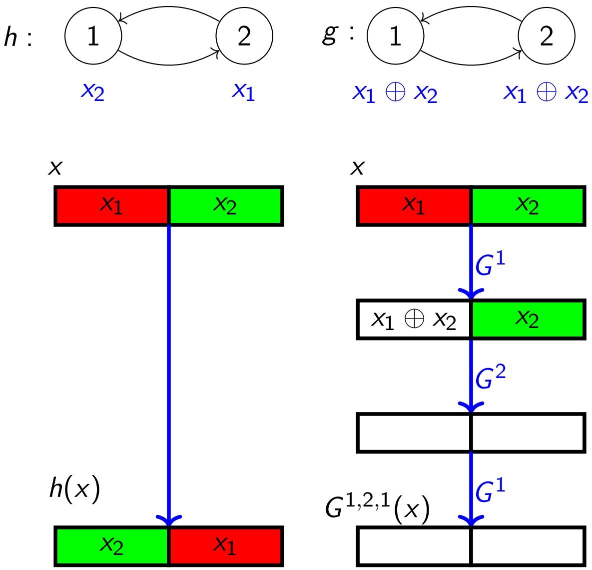

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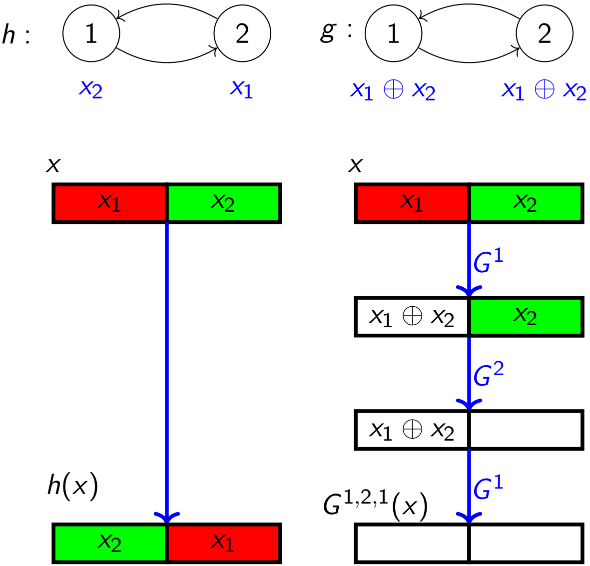

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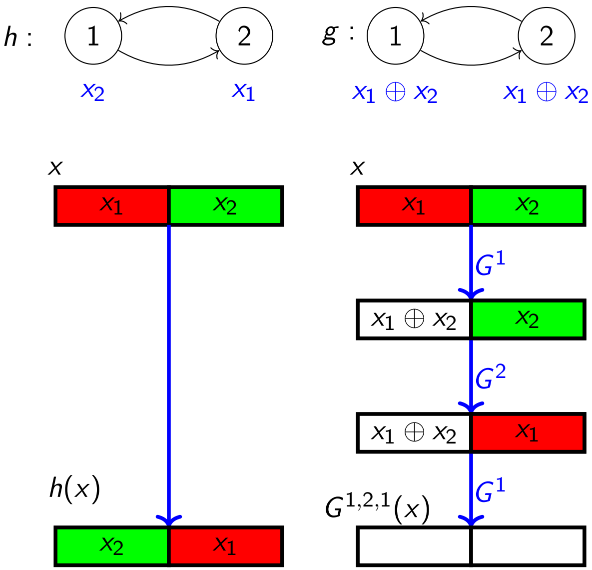

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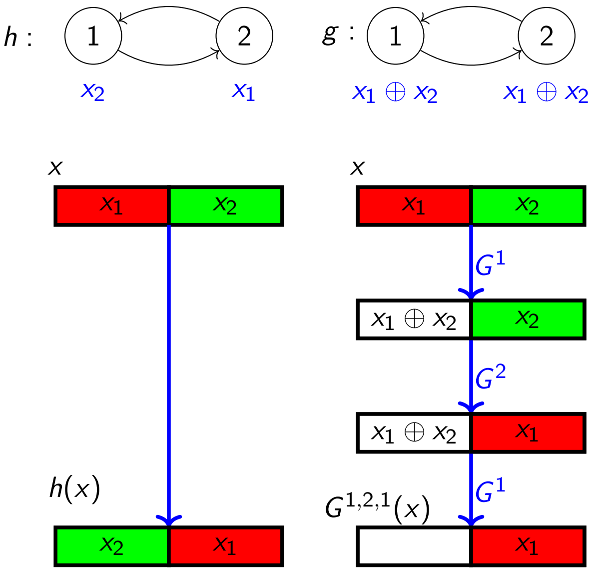

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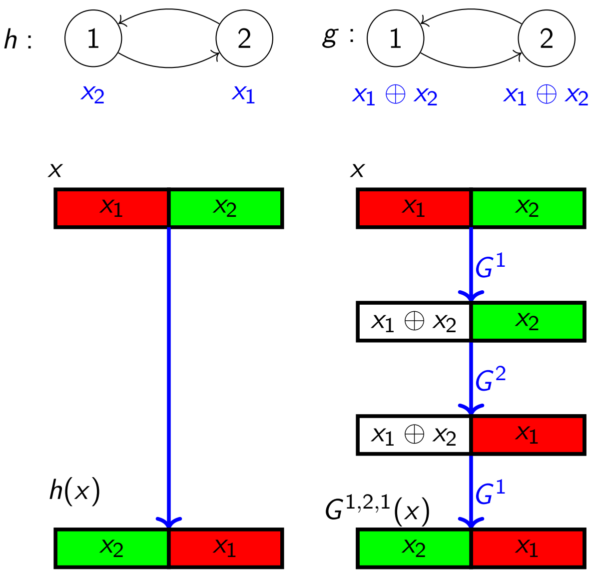

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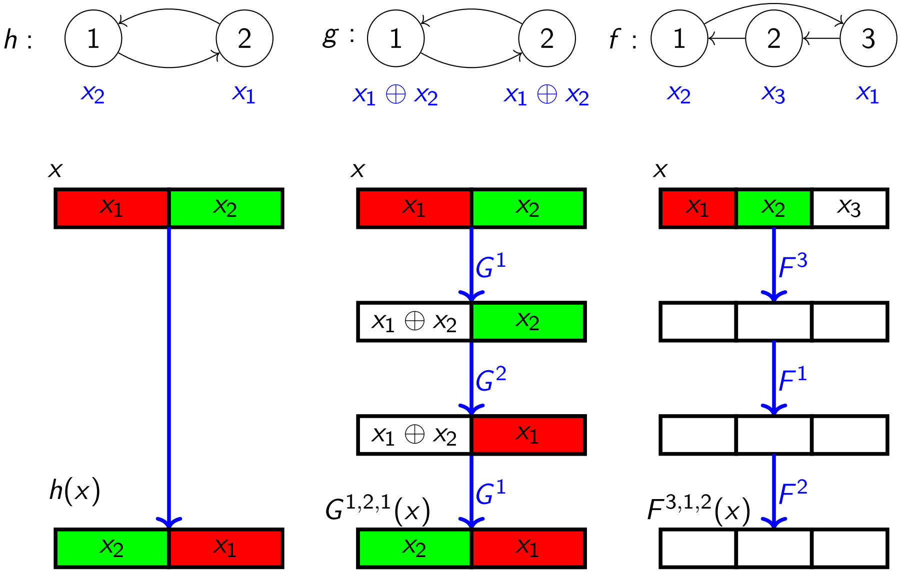

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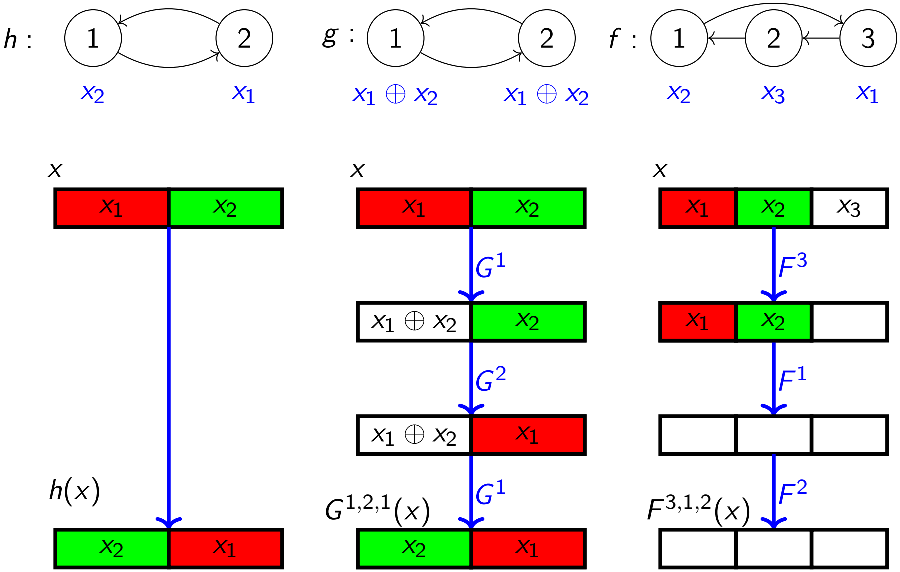

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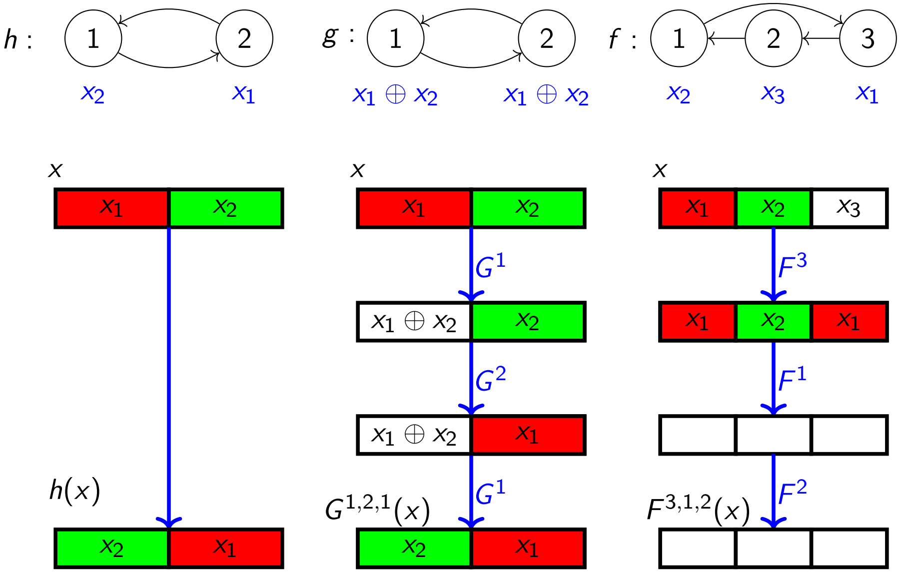

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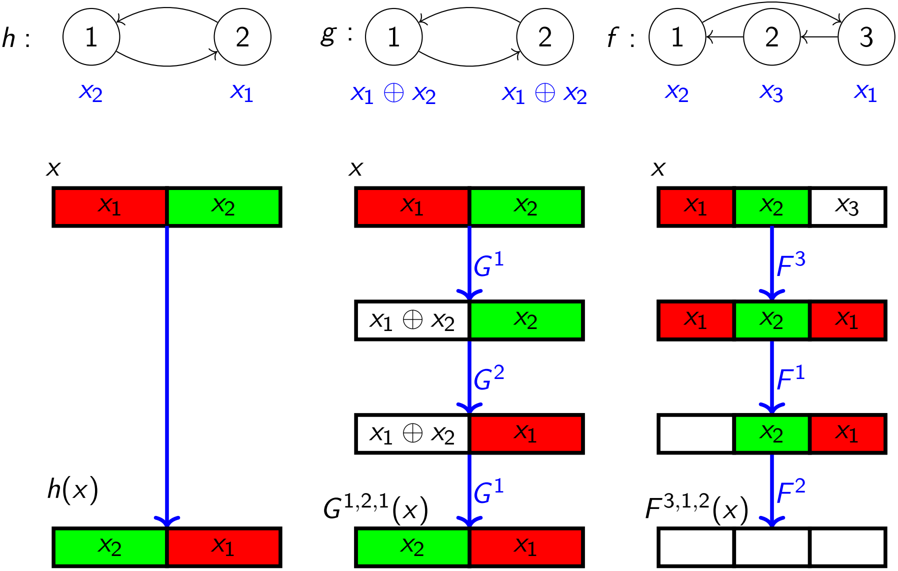

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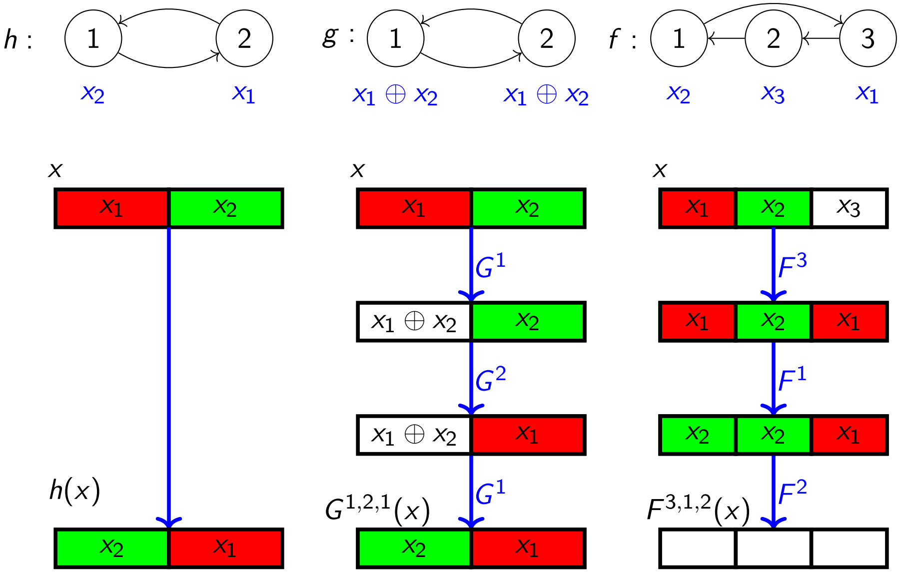

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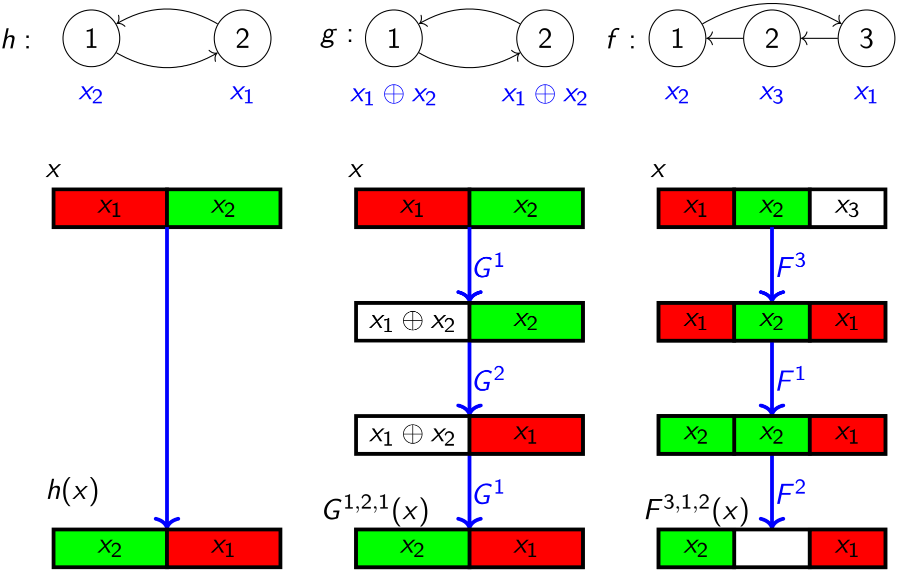

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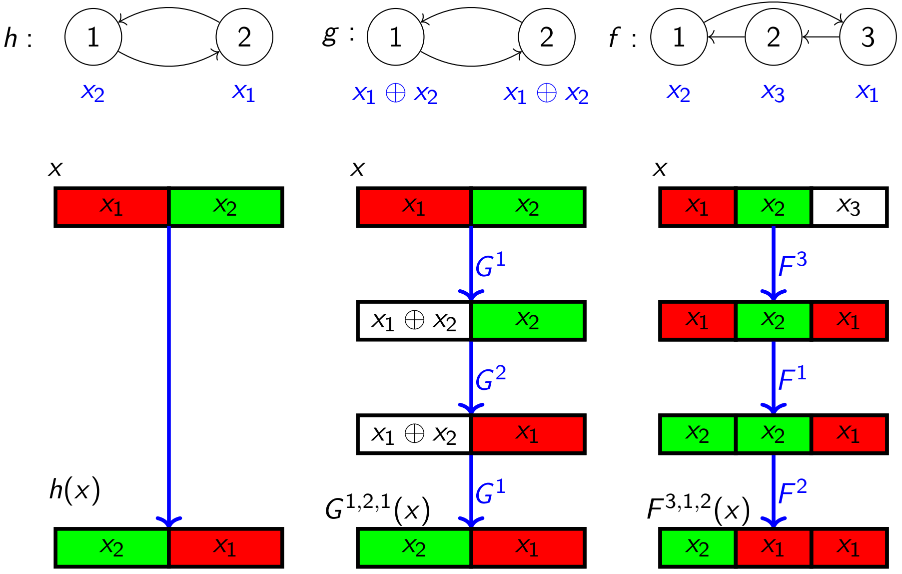

N-completeness

Definition (n-complete)

$f \in \mathrm{Tran}(A^m)$ is n-complete if $\forall h \in A^n$, $\exists t \geq 0$, $\exists w \in [n]^t$, $F^w \triangleright h$.

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$$(x_1, x_2) \xrightarrow{F^2} (x_1, x_1) \xrightarrow{F^1} (\neg x_1, x_1) \xrightarrow{F^1} (1, x_1) \xrightarrow{F^2} (1, 1) \xrightarrow{F^1} (0, 1).$$

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Complete Automata networks with optimized time

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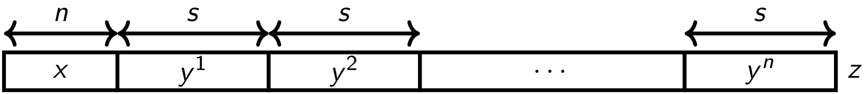

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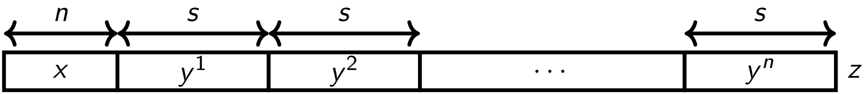

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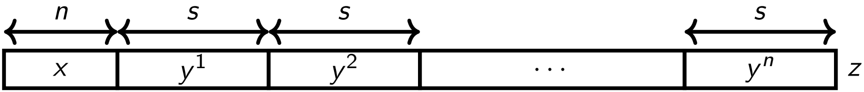

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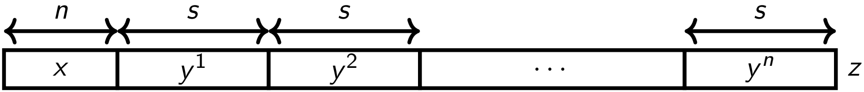

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$$F^{(1,i_1),(2,i_2),\dots,(n,i_n),1,\dots,n} \triangleright h$$

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$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

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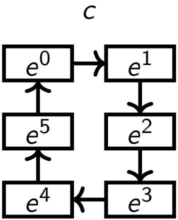

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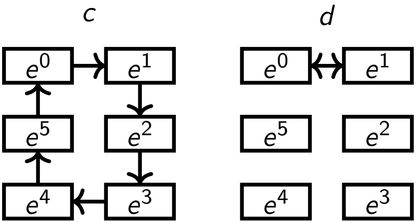

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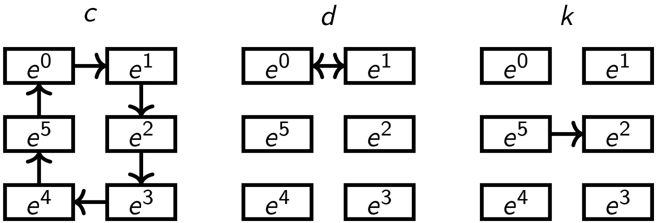

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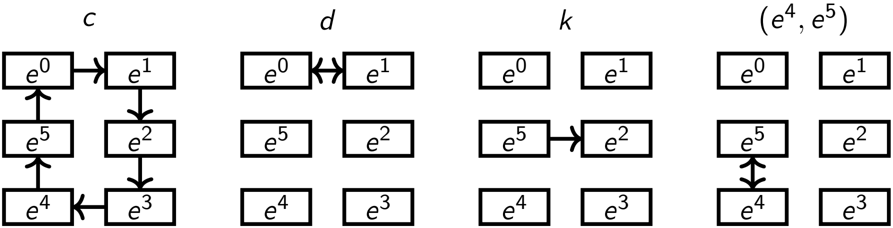

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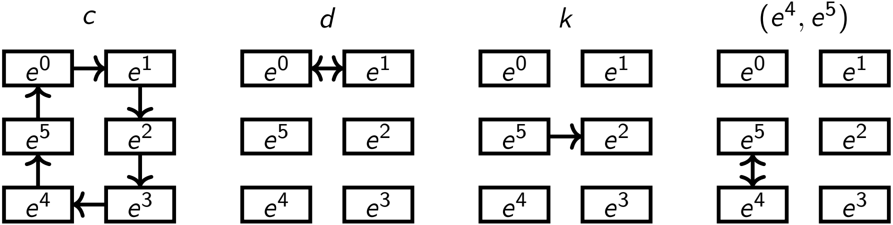

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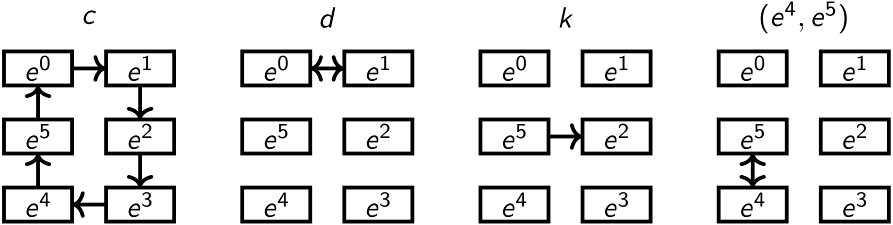

e	e^0	e^1	e^2	e^3	e^4	e^5
$c(e)$	e^1	e^2	e^3	e^4	e^5	e^0

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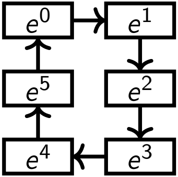
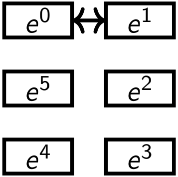
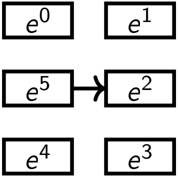
c



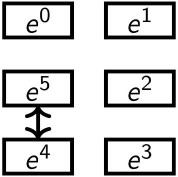
d



k

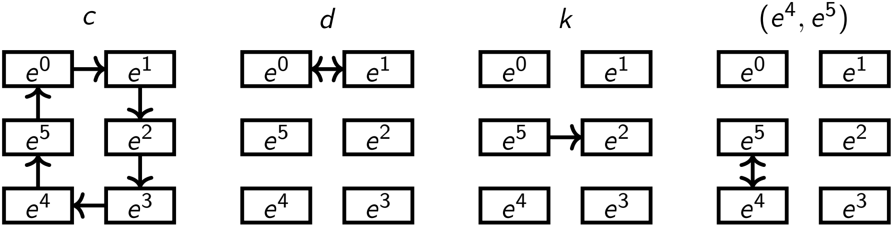


(e^4, e^5)



e	e^0	e^1	e^2	e^3	e^4	e^5
$c^2(e)$	e^2	e^3	e^4	e^5	e^0	e^1

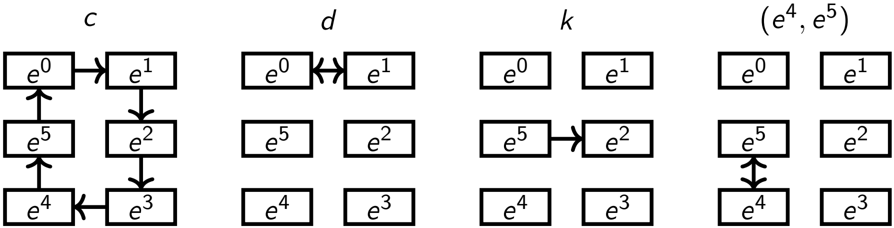
$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



e	e^0	e^1	e^2	e^3	e^4	e^5
$(d \circ c^2)(e)$	e^2	e^3	e^4	e^5	e^1	e^0

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

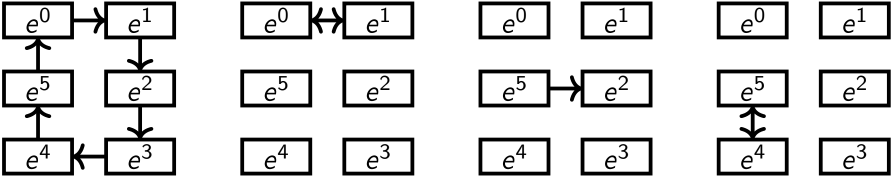


e	e^0	e^1	e^2	e^3	e^4	e^5
$(c \circ d \circ c^2)(e)$	e^3	e^4	e^5	e^0	e^2	e^1

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

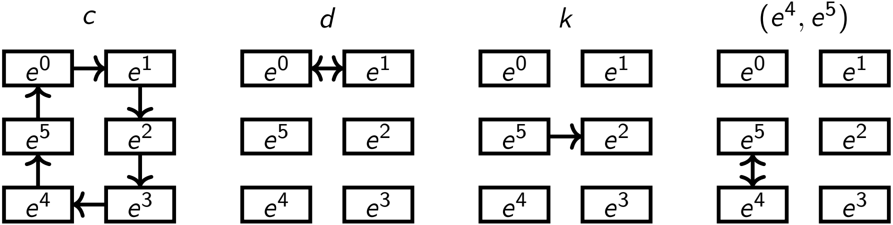
c $\quad$ d $\quad$ k $\quad$ (e^4, e^5)



e	e^0	e^1	e^2	e^3	e^4	e^5
$(c^2 \circ d \circ c^2)(e)$	e^4	e^5	e^0	e^1	e^3	e^2

Generating set

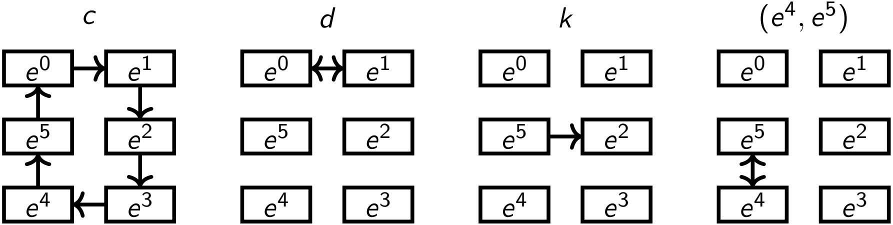
$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



e	e^0	e^1	e^2	e^3	e^4	e^5
$(c^3 \circ d \circ c^2)(e)$	e^5	e^0	e^1	e^2	e^4	e^3

Generating set

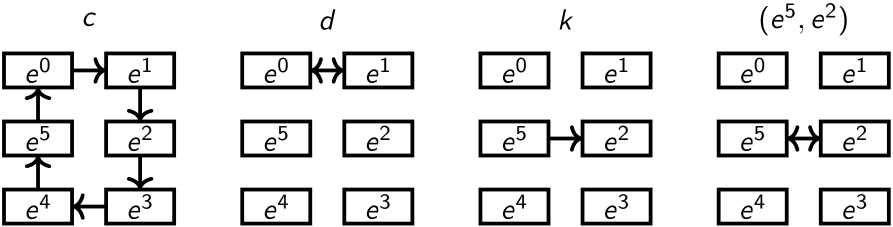
$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



e	e^0	e^1	e^2	e^3	e^4	e^5
$(c^4 \circ d \circ c^2)(e)$	e^0	e^1	e^2	e^3	e^5	e^4

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

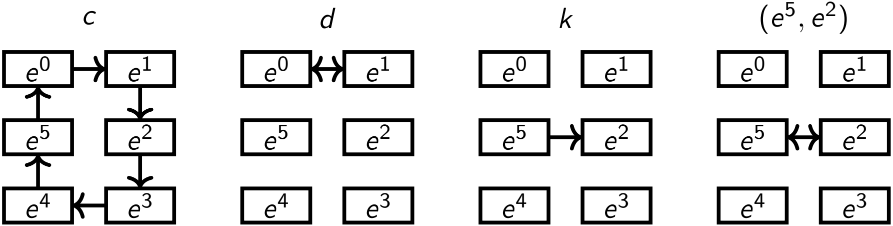


e	e^0	e^1	e^2	e^3	e^4	e^5
e	e^0	e^1	e^2	e^3	e^4	e^5

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

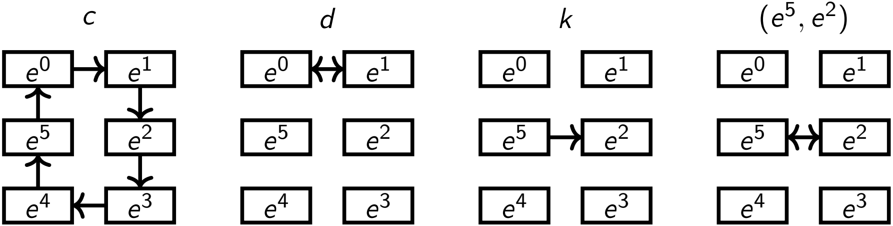


e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^3, e^2)(e)$	e^0	e^1	e^3	e^2	e^4	e^5

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$

Generating set

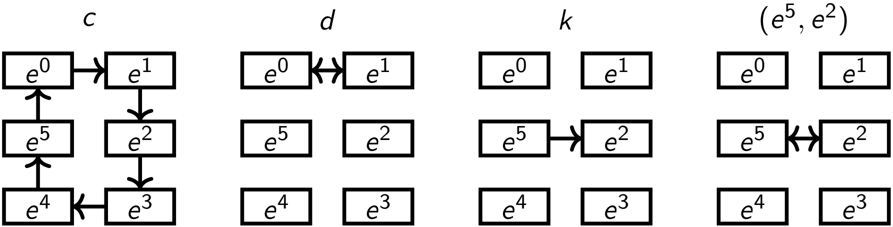
$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^4, e^3)(e^3, e^2)(e)$	e^0	e^1	e^4	e^2	e^3	e^5

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$

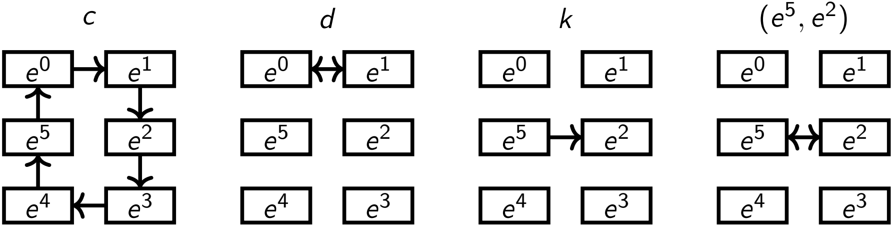
$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^5, e^4)(e^4, e^3)(e^3, e^2)(e)$	e^0	e^1	e^5	e^2	e^3	e^4

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$

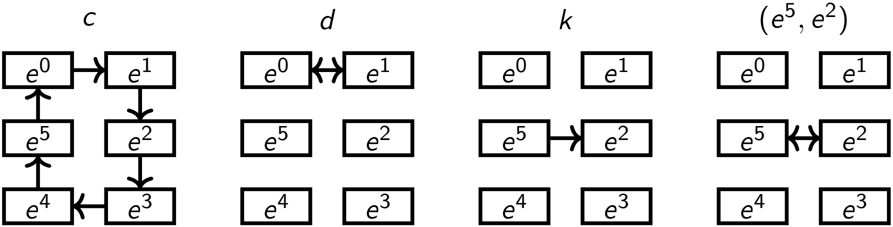
$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^3, e^4)(e^5, e^4)(e^4, e^3)(e^3, e^2)(e)$	e^0	e^1	e^5	e^2	e^4	e^3

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$

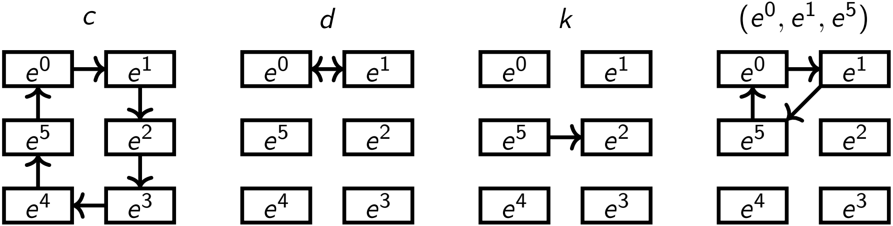
$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^2, e^3)(e^3, e^4)(e^5, e^4)(e^4, e^3)(e^3, e^2)(e)$	e^0	e^1	e^5	e^3	e^4	e^2

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

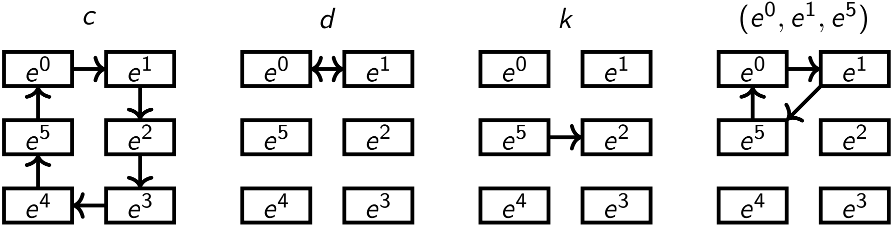


e	e^0	e^1	e^2	e^3	e^4	e^5
e	e^0	e^1	e^2	e^3	e^4	e^5

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



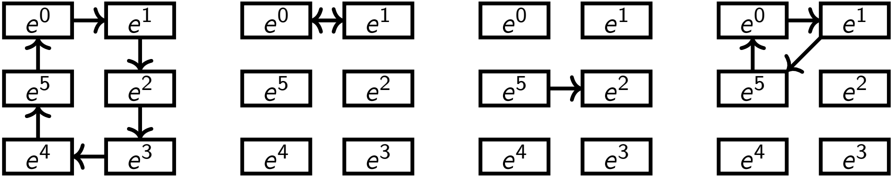
e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^5, e^0)(e)$	e^5	e^1	e^2	e^3	e^4	e^0

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

c $\qquad$ d $\qquad$ k $\qquad$ (e^0, e^1, e^5)

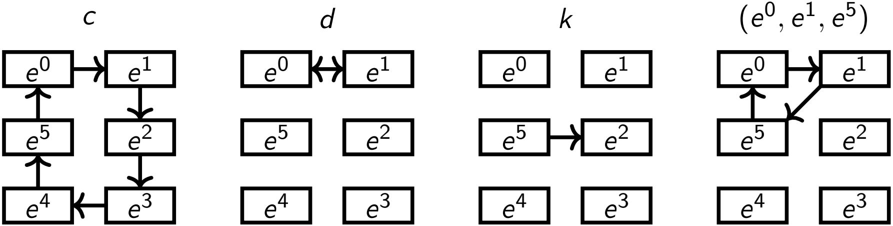


e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^1, e^5)(e^5, e^0)(e)$	e^1	e^5	e^2	e^3	e^4	e^0

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

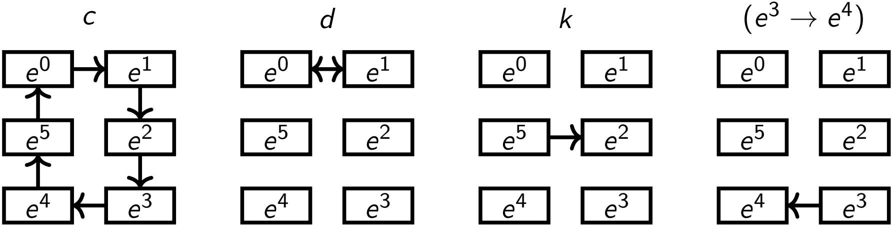


e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^0, e^1)(e^1, e^5)(e^5, e^0)(e)$	e^0	e^5	e^2	e^3	e^4	e^1

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

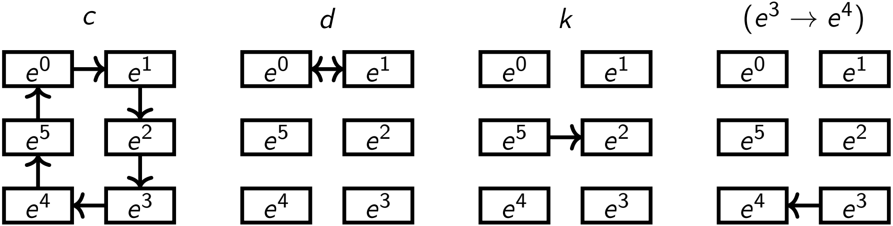


e	e^0	e^1	e^2	e^3	e^4	e^5
e	e^0	e^1	e^2	e^3	e^4	e^5

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

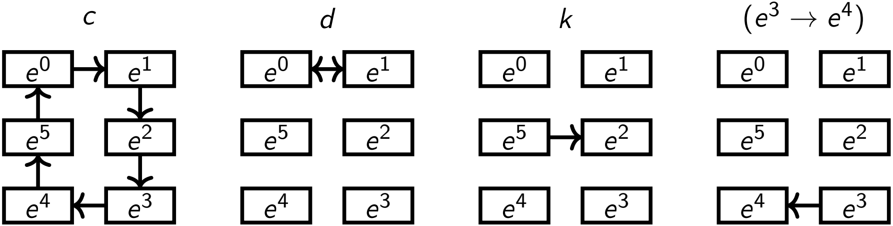


e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^3, e^5)e$	e^0	e^1	e^2	e^5	e^4	e^3

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

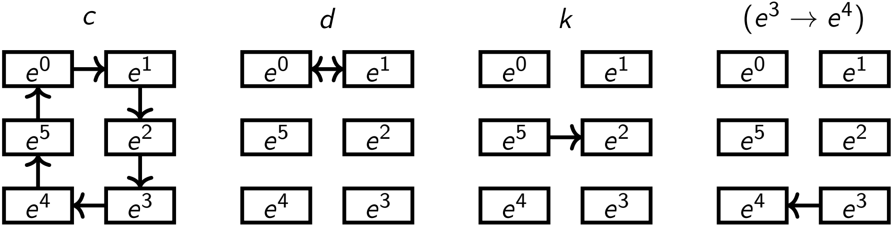


e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^4, e^2)(e^3, e^5)e$	e^0	e^1	e^4	e^5	e^2	e^3

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

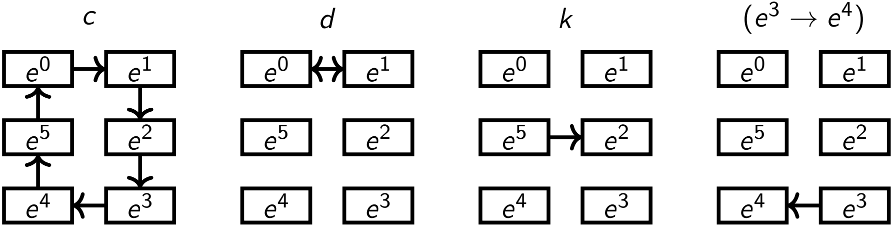


e	e^0	e^1	e^2	e^3	e^4	e^5
$k(e^4, e^2)(e^3, e^5)e$	e^0	e^1	e^4	e^2	e^2	e^3

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

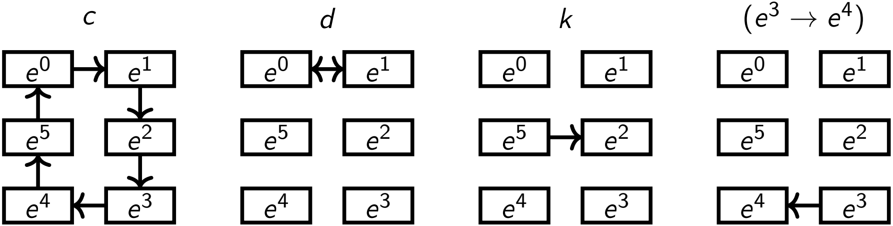


e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^3, e^5)k(e^4, e^2)(e^3, e^5)e$	e^0	e^1	e^4	e^2	e^2	e^5

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.

Generating set

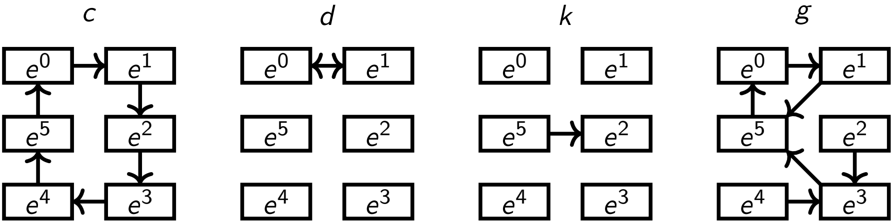
$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^4, e^2)(e^3, e^5)k(e^4, e^2)(e^3, e^5)e$	e^0	e^1	e^2	e^4	e^4	e^5

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.
- $(e^i \to e^j) = (e^j, e^2) \circ (e^i, e^5) \circ k \circ (e^j, e^2) \circ (e^i, e^5)$.

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

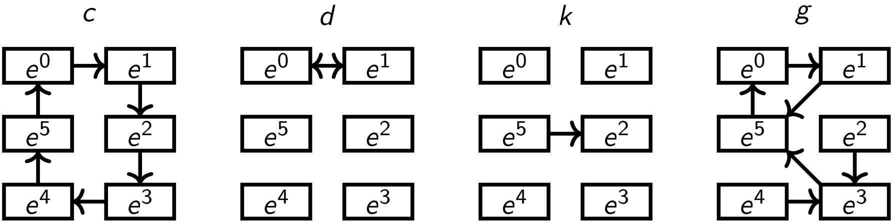


e	e^0	e^1	e^2	e^3	e^4	e^5
e	e^0	e^1	e^2	e^3	e^4	e^5

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.
- $(e^i \to e^j) = (e^j, e^2) \circ (e^i, e^5) \circ k \circ (e^j, e^2) \circ (e^i, e^5)$.

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

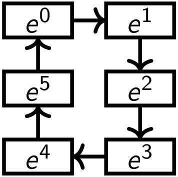


e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^0, e^1, e^5)e$	e^1	e^5	e^2	e^3	e^4	e^0

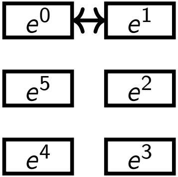
- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \dots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \dots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \dots, e^{i_p}) = (e^1, e^{i_p}) \circ \dots \circ (e^{i_2}, e^{i_1})$.
- $(e^i \to e^j) = (e^j, e^2) \circ (e^i, e^5) \circ k \circ (e^j, e^2) \circ (e^i, e^5)$.

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

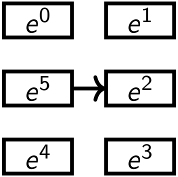
c



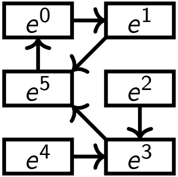
d



k



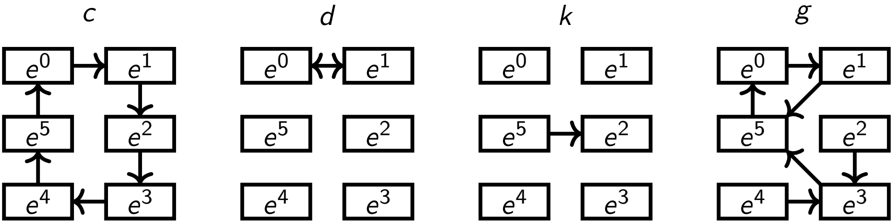
g



e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^3 \to e^5)(e^0, e^1, e^5)e$	e^1	e^5	e^2	e^5	e^4	e^0

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.
- $(e^i \to e^j) = (e^j, e^2) \circ (e^i, e^5) \circ k \circ (e^j, e^2) \circ (e^i, e^5)$.

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$

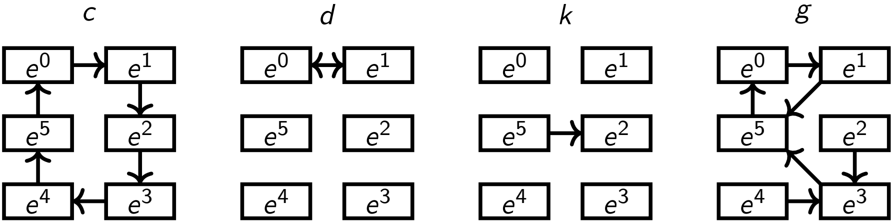


e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^2 \to e^3)(e^3 \to e^5)(e^0, e^1, e^5)e$	e^1	e^5	e^3	e^5	e^4	e^0

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \dots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \dots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \dots, e^{i_p}) = (e^1, e^{i_p}) \circ \dots \circ (e^{i_2}, e^{i_1})$.
- $(e^i \to e^j) = (e^j, e^2) \circ (e^i, e^5) \circ k \circ (e^j, e^2) \circ (e^i, e^5)$.

Generating set

$$E := \{e^0, e^1, e^2, e^3, e^4, e^5\}$$



e	e^0	e^1	e^2	e^3	e^4	e^5
$(e^4 \to e^3)(e^2 \to e^3)(e^3 \to e^5)(e^0, e^1, e^5)e$	e^1	e^5	e^3	e^5	e^3	e^0

- $(e^i, e^{i+1}) = c^i \circ d \circ c^{|E|-i}$
- $(e^i, e^j) = (e^i, e^{i+1}) \circ (e^{i+1}, e^{i+2}) \circ \cdots \circ (e^{j-1}, e^j) \circ (e^{j-1}, e^{j-2}) \circ \ldots (e^{i+1}, e^i)$
- $(e^{i_1}, e^{i_2}, \ldots, e^{i_p}) = (e^1, e^{i_p}) \circ \cdots \circ (e^{i_2}, e^{i_1})$.
- $(e^i \to e^j) = (e^j, e^2) \circ (e^i, e^5) \circ k \circ (e^j, e^2) \circ (e^i, e^5)$.

Complete Automata networks with optimized size

$$f_{n+1}(x) := x_1$$

$$f_1(x) := \begin{cases} x_1 + 1 & \text{if } x_1 = x_{n+1} \\ 0 & \text{if } x_1 \neq x_{n+1} \text{ and } x_{[n]} = (0)^n \\ 1 & \text{if } x_1 \neq x_{n+1} \text{ and } x_{[n]} = 1(0)^{n-1} \\ x_1 & \text{otherwise} \end{cases}$$

$$f_2(x) := \begin{cases} x_2 + 1 & \text{if } x_1 \neq x_{n+1} \text{ and } x_1 = 0 \\ 1 & \text{if } x_1 = x_{n+1} \text{ and } x_{[n]} = 0^n \\ x_1 & \text{otherwise} \end{cases}$$

$$\forall i \in [3, n], f_i(x) := \begin{cases} x_i + 1 & \text{if } x_1 = x_2 = \cdots = x_{i-1} = 0 \\ x_i & \text{otherwise} \end{cases}$$

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$$F^{n+1,1,2,\ldots,n} \triangleright ((0,0,\ldots), \ldots, (q-1,0,\ldots), (0,1,\ldots), (1,1,\ldots), \ldots).$$

Complete Automata networks with optimized size

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$$F^{n+1,1,2,\dots,n} \triangleright ((0,0,\dots),\dots,(q-1,0,\dots),(0,1,\dots),(1,1,\dots),\dots).$$

$$F^{(n+1,1)^q,1} \triangleright (0^n, 10^{n-1}).$$

Complete Automata networks with optimized size

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$$F^{(n+1,1)^q,1} \triangleright (0^n, 10^{n-1}).$$

$$F^{n+1,2} \triangleright (0^n \to 010^{n-2}).$$

Conclusion and ongoing work

- We have seen two constructions of n-complete function,
  - One of time $2n$ (but with a big size).
  - One of size $n+1$ (but with a big time).
- We are currently looking for other related problems,
  - We try to use a bigger alphabet. We have found some functions $f \in \mathrm{Tran}(B^n)$ which can simulate any $h \in \mathrm{Tran}(A^n)$. Depending on the definition of simulation, we have to take $|B| = 2|A|$ or $|B| = |A| + 1$.
  - We try to characterize the kind of transformation that can't be simulated by a transformation of same size and alphabet.