Combinatoric Analysis for Cardinality Constraints

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Summary

Constraint Programming

Constraint Programming is used to solve satisfaction and optimization problems.

Constraint Satisfaction Problem :

- A set of variables $X = \{x_1, \dots, x_n\}$
- Each variable x_i can only take a value from a domain D_i (**discrete**)
- A set of constraints $C = \{C_1, \dots, C_p\}$, which are logical connections between variables.

Applications : Scheduling problems, assignment problems, transportation problems, stock management, etc...

The *alldifferent* constraint

There are different types of constraint :

- Arithmetic : $<, >, \leq, \geq, =, \neq$
- On graphs : *circuit*, *tree*
- On words : *regular*
- Cardinality Constraints : *alldifferent*, *global_cardinality*, *nvalue*

alldifferent [4]

The constraint $alldifferent(x_1, \ldots, x_n)$ ensures that every variable takes different value.

Example : The n-queens problem

The model :

- $X = \{x_1, \ldots, x_n\}$, a variable = a column
- $\forall x_i \in X, D_i = \{1, \ldots, n\}$, a value = a row
- Constraints :
  - every queen must be on different rows : $alldifferent(x_1, \ldots, x_n)$.
  - every queen must be on different ascending diagonals : $\forall i \neq j, x_i + i \neq x_j + j$
  - every queen must be on different descending diagonals : $\forall i \neq j, x_i + j \neq x_j + i$

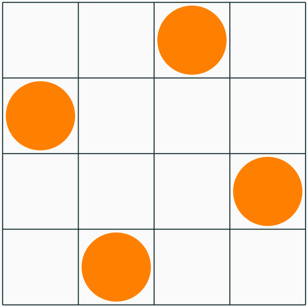
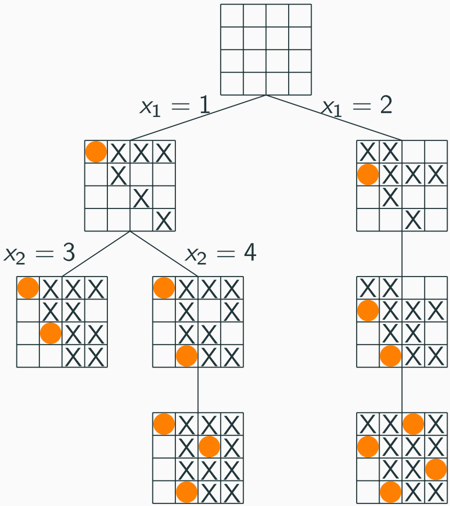


Figure 1 – A solution for $n = 4$: $(x_1 = 2, x_2 = 4, x_3 = 1, x_4 = 3)$

Example : The n-queens problem

Resolution :

- Instantiation Variable/Value
- Propagation
- Domain empty → backtrack



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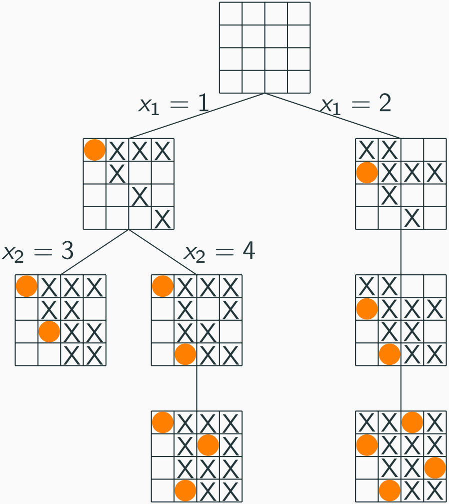
Which instantiation ?

We need heuristics



Combinatorics in Cardinality Constraints

$X = \{x_1, x_2, x_3, x_4\}$ $D_1 = \{1, 2, 4\}$,
$D_2 = \{2, 3\}$, $D_3 = \{1, 2, 3, 5\}$, $D_4 = \{4, 5\}$

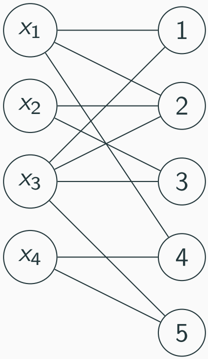


Figure 2 – Model for *alldifferent*

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One solution = One matching covering X

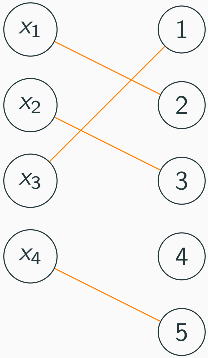


Figure 2 – Model for *alldifferent*

alldifferent : Each value y_j must be assigned at most once

global_cardinality [5] : Each value y_j must be assigned at least l_j times and at most u_j times

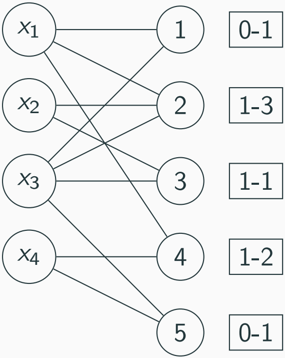
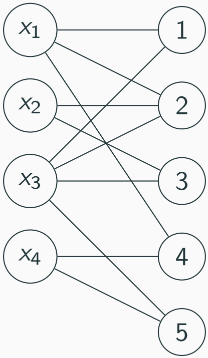


Figure 3 – Example of an instance of *global_cardinality*

Biadjacency matrix :

$$B = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$



$$|X| = n \leq |Y| = m$$

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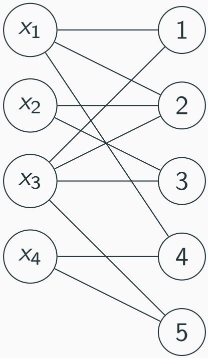


$$|X| = n \leq |Y| = m$$

Permanent

$$Perm(B) = \frac{1}{(m-n)!} \sum_{\sigma \in \mathfrak{S}_m} \prod_{i=1}^{n} b_{i\sigma(i)}$$

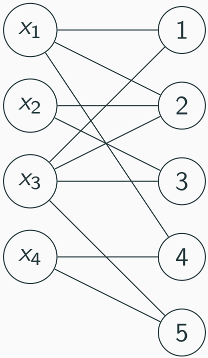
There are $Perm(B) = 13$ solutions in our example.

Counting the number of matchings in a bipartite graph is #P-Complete [6]
We can compute a bound in polynomial time :

Brégman-Minc upper bound [1]

$$Perm(B) \leq \prod_{i=1}^{n} (d_i!)^{\frac{1}{d_i}}$$

with $d_i = \sum_{j=1}^{m} b_{ij}$

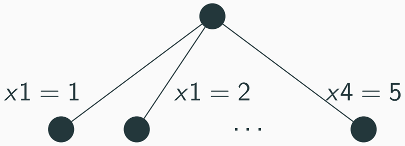


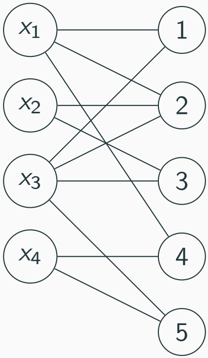
$|X| = n \leq |Y| = m$

Counting Solutions on *alldifferent*

Counting-Based Search [3]

We first explore the sub-problem where there are likely most solutions





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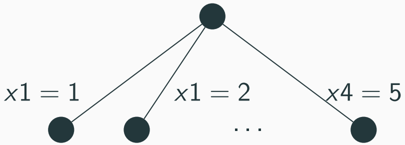

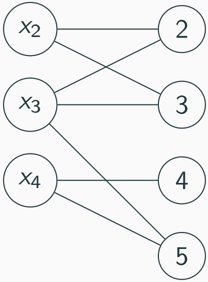

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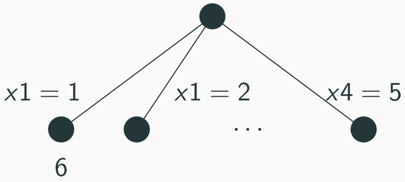

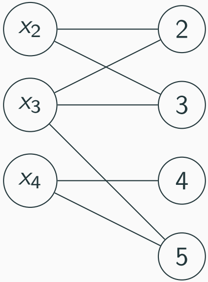

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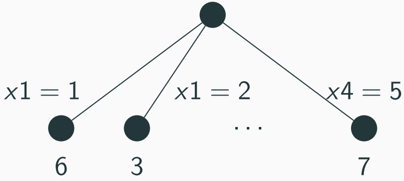

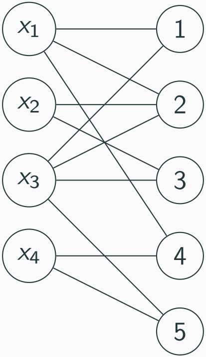

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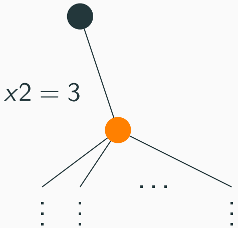

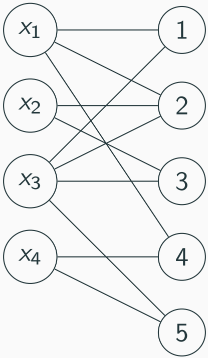

Probabilistic model for *alldifferent*

Erdős-Renyi Model [2]

$\forall x_i \in X, \forall y_j \in Y, \mathbb{P}(\{y_j \in D_i\}) = p \in [0, 1]$ and $\{y_j \in D_i\}$ independent

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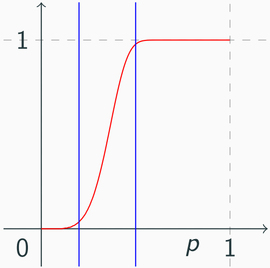

Figure 4 – Phase transition for the existence of a solution

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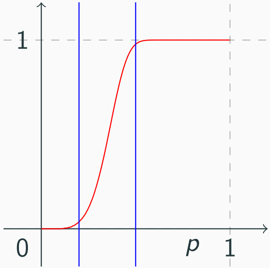
Example

For our example, we can take $p = \frac{11}{20}$:

- We can expect $10,98$ solutions
- We are almost sure that there is a solution



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Questions ?